Generalized Cournot-Stackelberg Models

An appendix to *Pass-through as an Economic Tool*

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I. Vertical monopolies results

The natural method for solving this problem is backward induction. Firms in group k take the mark-ups chosen by all firms in groups weakly above them as given and, conjecturing a particular equilibrium, take the mark-up of all firms strictly below them as a function of the mark-up they choose. Let the conjectured total mark-up of all firms in lower groups be $\overline{M}_{k-1}\left(c + \sum_{k'=k}^{K} \sum_{i=1}^{N_{k'}} m_{i_{k'}}\right)$. Then the anticipated profits each firm in round k is

$$\pi_{i_k}(m_{i_k}) = m_{i_k} Q\left(c + \sum_{k'=k}^{K} \sum_{i'=1}^{N_{k'}} m_{i'_{k'}} + \overline{M}_{k-1}\left[c + \sum_{k'=k}^{K} \sum_{i'=1}^{N_{k'}} m_{i'_{k'}}\right]\right) \tag{14}$$

If we let $c_k \equiv c + \sum_{k'=k+1}^{K} \sum_{i'=1}^{N_{k'}} m_{i'_{k'}}$ and $Q_k(p) \equiv Q\left(p + \overline{M}_{k-1}[p]\right)$ then we can rewrite equation (14) as

$$\pi_{i_k}(m_{i_k}) = m_{i_k} Q_k\left(c_k + \sum_{i'=1}^{N_k} m_{i'_k}\right)$$

which is simply the profit function for (any) one firm in the simultaneous vertical monopolies problem with cost c_k and demand Q_k. It is well-known that the conditions for a symmetric interior equilibrium in this game are

$$\overline{m}_k^\star = N_k \mu_k(c_k + \overline{m}_k^\star) \tag{15}$$

$$m_{i_k}^\star = \frac{\overline{m}_k^\star}{N_k} \quad \forall i = 1, \ldots, N_k \tag{16}$$

where $\mu_k(p) \equiv -\frac{Q_k(p)}{Q_k'(p)}$ and $\overline{m}_k^\star \equiv \sum_{i'=1}^{N_k} m_{i'_k}$. In fact these conditions identify the unique equilibrium of the game (which is stable) for all values of c_k if (and essentially only if) $\mu_k'(p) < \frac{1}{N_k}$ wherever $Q_k(p)$ is strictly positive (Seade, 1980). Furthermore any equilibrium not given by equations (15) and (16) at a point where $\mu_k'(\overline{m}_k) < \frac{1}{N_k}$ is unstable or on a boundary. This condition holding for all k I refer to as *GCS stability*. Finally note that m_{i_k} is the same for all i_k, so I will denote by $m_k^\star$ the equilibrium mark-up of any firm at level k. Before formally stating the results given in the text, I establish a lemma I use in the proof of one of the results, which provides a useful alternative characterization of the solution to the GCS model.

Lemma 2. *Let $\overline{\rho}_0 \equiv 1$ and let $\overline{\mu}_0 \equiv 0$. Then for $k = 1, \ldots, K$ let $(\overline{\mu}_k, \overline{\rho}_k)$ be defined inductively by $\overline{\mu}_k \equiv \frac{N_k \mu}{\overline{\rho}_{k-1}} + \overline{\mu}_{k-1}$ and $\overline{\rho}_k \equiv \frac{1}{1-\overline{\mu}_k'}$. Then under GCS stability*

$$p^\star = \overline{\mu}_K(p^\star) + c \tag{17}$$

Proof. I prove inductively. The base case is given. Suppose that the result holds for the case of K. I must show it holds for $K+1$. By the inductive hypothesis the total mark-up of all firms at or below level K and below solves the equation

$$\overline{M}_K(c_K) = \overline{\mu}_K \left[\overline{M}_K(c_K) + c_K\right]$$

Therefore by the implicit function theorem

$$\overline{M}_K' = \frac{\overline{\mu}_K'}{1 - \overline{\mu}_K'}$$

The profits of firm i at stage $K+1$ are

$$m_{i_{K+1}} Q \left(c + \sum_{j=1}^{N_{K+1}} m_{j_{K+1}} + \overline{M}_K \left[c + \sum_{j=1}^{N_{K+1}} m_{j_{K+1}} \right] \right)$$

So her first-order condition is, dropping arguments

$$m_{i_{K+1}} Q'(1 + \overline{M}_K') + Q = 0$$

or

$$m_{i_{K+1}} = \frac{\mu}{\overline{\rho}_K}$$

thus equilibrium conditions are

$$\overline{m}_{K+1} = \frac{N_{K+1}\mu}{\overline{\rho}_K}$$

Thus the total price must solve

$$p^\star = c + \overline{m}_{K+1} + \overline{M}_K\left(c + \overline{m}_{K+1}\right) =$$

$$c+\overline{\mu}_K\left[c+\overline{m}_{K+1}+\overline{M}_K\left(c+\overline{m}_{K+1}\right)\right]+\frac{N_{K+1}\mu\left[c+\overline{m}_{K+1}+\overline{M}_K\left(c+\overline{m}_{K+1}\right)\right]}{\overline{\rho}_K\left[c+\overline{m}_{K+1}+\overline{M}_K\left(c+\overline{m}_{K+1}\right)\right]}=c+\overline{\mu}_{K+1}(p^\star)$$

I now formally state and prove the results given in the text.

Theorem 8. *Assume that ρ_k obeys SPAs for all k and assume GCS stability in all industrial organizations. Thus all statements below compare unique, stable equilibria across industrial organizations.*

1. *Fix any K and any strictly positive set of integers $\{N_k\}_{k<K}$. $m_K^\star$ is larger when N_K is larger (smaller) if and only if $\rho_K > (<)1$.*

2. *$m_{k+1}^\star > (<) m_k^\star \iff \rho_k < (>)1$.*

3. *In the situation described by the first result, if $N_K > 1$ then I can define an alternative organization where $\tilde{N}_k = N_k$ for all $k < K$, $\tilde{N}_K = N_K - 1$ and $\tilde{N}_{K+1} = 1$. Let $m_K^\star$ again be the equilibrium mark-up of a firm at level K in the original organization and let $\tilde{m}_{K+1}$ be the equilibrium mark-up of the one firm at level $K+1$ in the new industrial organization. $\tilde{m}_{K+1}^\star > (<) m_K^\star \iff \rho_K < (>)1$.*

4. *Fix any K and strictly positive integers $\{N_k\}_{k<K-1}$ and $N_K = 1$. Let $m_K^\star$ be the equilibrium mark-up of the firm in level K under this organization. Consider another organization with $\tilde{N}_k = N_k$ for all $k < K$, but with $N_K = \tilde{N}$ and $N_{K+1} = 1$. Let the mark-up of the leader in the new equilibrium be $\tilde{m}_{K+1}^\star$. $\tilde{m}_{K+1}^\star > (<) m_K^\star \iff \rho_K' < (>)0$ (in either organization, as they are the same).*

5. *Consider any industrial organization $(K, \{N_k\}_{k\leq K})$. Select a particular natural number $\tilde{k} \leq K$ for which $N_{\tilde{k}} > 1$. Define a second industrial organization by $\tilde{K} = K$, $\tilde{N}_k = N_k$ for all $k \neq \tilde{k}, \tilde{k}+1$, $\tilde{N}_{\tilde{k}} = N_{\tilde{k}} - 1$ and $\tilde{N}_{\tilde{k}+1} = N_{\tilde{k}+1} + 1$. Let the final price to consumer under the first organization be $p^\star$ and $\tilde{p}^\star$ under the second organization. $\tilde{p}^\star > (<) p^\star \iff (\rho_{\tilde{k}} - 1)(N_{\tilde{k}} - 1 - \rho_k N_{\tilde{k}+1}) < (>)0$.*

Proof. We go point by point.

1. By equations (15) and (16)

$$m_K^\star = \mu_K(c_K + \overline{m}_K^\star) = \mu_K(c + \overline{m}_K^\star)$$

Because the equilibrium is stable and $\mu_K > 0$, $\overline{m}_K^\star$ increases in N_K. Thus $m_K^\star$ is increasing (decreasing) in N_K if and only if $\mu_K' > (<)0$. But by implicit differentiation

$$\rho_K = \frac{dp_K}{dc} = \frac{1}{1 - N_K \mu_K'} \tag{18}$$

and thus $\mu_K' < (>)0 \iff \rho_K < (>)1$.

2. The profits of a firm at level $k+1$ are

$$m_{i_{k+1}} Q_k \left(c_{k+1} + \sum_{i' \neq i} m_{i'_{k+1}} + m_{i_{k+1}} + \overline{m}_k \left[c_{k+1} + \sum_{i' \neq i} m_{i'_{k+1}} + m_{i_{k+1}} \right] \right)$$

Thus the firm's first-order condition is

$$Q_k + Q_k' m_{i_{k+1}} \left(1 + \overline{m}_k'\right)$$

But clearly $1 + \overline{m}_k' = \rho_k$ and thus I can rewrite this as

$$m_{i_{k+1}}^\star = \frac{\mu_k}{\rho_k}$$

which is clearly greater (less) than $m_k^\star = \mu_k$ if $\rho_k < (>)1$.

3. By the proof of the second result $\tilde{m}_{K+1}^\star = \frac{\mu_K(c + \tilde{m}_{K+1}^\star + [N_K - 1]\tilde{m}_K^\star)}{\rho_K(c + \tilde{m}_{K+1}^\star + [N_K - 1]\tilde{m}_K^\star)}$ and therefore

$$\tilde{m}_{K+1} + (N_K - 1)\tilde{m}_K^\star = \mu_K(c + \tilde{m}_{K+1}^\star + [N_K - 1]\tilde{m}_K^\star) \left(N_K - 1 + \frac{1}{\rho_K} \right)$$

or if I define $\tilde{m}^\star \equiv \tilde{m}_{K+1} + (N_K - 1)\tilde{m}_K^\star$ I have

$$\tilde{m}^\star = \mu_K(c + \tilde{m}^\star) \left(N_K - 1 + \frac{1}{\rho_K} \right)$$

Therefore, again by stability (and the SPAs),

$$\tilde{m}_{K+1}^\star + (N_K - 1)\tilde{m}_K^\star = \tilde{m}^\star > (<)\overline{m}_K^\star = N_K m_K^\star \iff \rho_K < (>)1$$

By the second result I know that $\tilde{m}_{K+1} > (<)\tilde{m}_K^\star \iff \rho_K < (>)1$. Thus if and only if $\rho_K < (>) 1$

$$N_K \tilde{m}^\star_{K+1} > (<) \tilde{m}^\star_{K+1} + (N_K - 1)\tilde{m}^\star_K > (<) N_K m^\star_K$$

establishing the result.

4. I know that $m^\star_K = \mu_K(c + m^\star_K)$ and that

$$\tilde{m}^\star_{K+1} = \mu_K(c + \tilde{m}^\star_{K+1} + \tilde{N}\tilde{m}^\star_K)\left(1 - \tilde{N}\mu'_K[c + \tilde{m}^\star_{K+1} + \tilde{N}\tilde{m}^\star_K]\right)$$

Thus $m^\star_K$ solves $m = \mu_K(c+m) \equiv f(m)$ and $\tilde{m}^\star_{K+1}$ solves

$$m = \mu_K\left(c + m + \tilde{N}m_K(m)\right)\left(1 - \tilde{N}\mu'_K[c + m + \tilde{N}m_K(m)]\right) \equiv g(m)$$

where m_K is the equilibrium mark-up of firms at K for cost level $c+m$, that is the (unique) solution to $m_K = \mu_K(m+c+\tilde{N}m_K)$. Note that by stability if $g(m) > (<) f(m)$ for all m then this implies that $\tilde{m}^\star_{K+1} > (<) m^\star_K$.

$$f(m) - g(m) = \mu_K(c+m) - \mu_K\left(c + m + \tilde{N}m_K(m)\right)\left(1 - \tilde{N}\mu'_K[c + m + \tilde{N}m_K(m)]\right)$$

Letting $\overline{p}(m) \equiv c + m + \tilde{N}m_K(m)$ and $\underline{p}(m) \equiv c + m$ allows me to rewrite this as

$$\mu_K(\underline{p}[m]) - \mu_K(\overline{p}[m])\left(1 - \tilde{N}\mu'_K[\overline{p}(m)]\right)$$

This has the same sign as

$$\frac{\mu_K\left(\underline{p}[m]\right)}{\mu_K\left(\overline{p}[m]\right)} - 1 + \tilde{N}\mu'_K\left(\overline{p}[m]\right) = \frac{\mu_K\left(\overline{p}[m]\right) + \int_{\overline{p}}^{\underline{p}} \mu'_K(p)dp}{\mu_K\left(\overline{p}[m]\right)} - 1 + \tilde{N}\mu'_K\left(\overline{p}[m]\right) =$$

$$\tilde{N}\mu'_K(\overline{p}[m]) - \frac{\int_{\underline{p}}^{\overline{p}} \mu'_K(p)dp}{\mu_K\left(\overline{p}[m]\right)} = \tilde{N}\mu'_K(\overline{p}[m]) - \frac{\overline{\mu'_K}\Big|_{\underline{p}}^{\overline{p}}\left(\overline{p} - \underline{p}\right)}{\mu_K\left(\overline{p}[m]\right)} = \tilde{N}\mu'_K(\overline{p}[m]) - \frac{\overline{\mu'_K}\Big|_{\underline{p}}^{\overline{p}}\tilde{N}m_K(m)}{\mu_K\left(\overline{p}[m]\right)}$$

where $\overline{f}(x)\big|_{x_0}^{x_1}$ is the average value of f over the range x_0 to x_1. Recall that by definition $\mu_K(\overline{p}[m]) = m_K(m)$. This lets me rewrite the expression as

$$\tilde{N}\left(\mu'_K[\overline{p}(m)] - \overline{\mu'_K}\Big|_{\underline{p}}^{\overline{p}}\right)$$

$\rho_K' < (>)0 \iff \mu_K'' < (>)0$ as $\rho_K = \frac{1}{1-\tilde{N}\mu_K'}$; note this also holds in the original organization as in this case $\rho_K = \frac{1}{1-\mu_K'}$. Thus clearly $\mu_K'[\overline{p}(m)] > (<) \left.\overline{\mu_K'}\right|_{\underline{p}}^{\overline{p}} \iff \rho_K' > (<)0$. Thus $\tilde{m}_{K+1}^\star > (<) m_K^\star \iff \rho_K' < (>)0$. Note this shows a sense in which only average behavior over the relevant range matters: the SPAs are a bit of overkill.

5. Note that by stability (positive pass-through), the final price to consumers is monotone increasing in $p_{\tilde{k}} = c_{\tilde{k}-1}$. Our approach is therefore to show that $\tilde{p}_{\tilde{k}}^\star$, the equilibrium price up through level $\tilde{k}$ under the second organization, is greater (less) than $p_{\tilde{k}}^\star$, the equilibrium price up through level $\tilde{k}$ under the first organization. To do this I use Lemma 2. According to the formulae there, $p_{\tilde{k}}^\star$ solves

$$p_{\tilde{k}}^\star = c_{\tilde{k}+1} + \left(N_{\tilde{k}} + \frac{N_{\tilde{k}+1}}{1 - N_{\tilde{k}}\mu_{\tilde{k}}'(p_{\tilde{k}}^\star)}\right)\mu_{\tilde{k}}(p_{\tilde{k}}^\star)$$

while

$$\tilde{p}_{\tilde{k}}^\star = c_{\tilde{k}+1} + \left(N_{\tilde{k}} - 1 + \frac{N_{\tilde{k}+1} + 1}{1 - (N_{\tilde{k}} - 1)\mu_{\tilde{k}}'(p_{\tilde{k}}^\star)}\right)\mu_{\tilde{k}}(p_{\tilde{k}}^\star)$$

Note that if I can show that, for any value of $c_{\tilde{k}+1}$ one expression is greater than the other, then this immediately implies a comparison between their equilibrium values by stability. More details of these arguments are available on request. Thus it is sufficient (plugging in the definition of $\rho_{\tilde{k}}$), again by stability, to establish a ranking between

$$N_{\tilde{k}} + \rho_{\tilde{k}} N_{\tilde{k}+1}$$

and

$$N_{\tilde{k}} - 1 + \frac{(N_{\tilde{k}+1} + 1)N_{\tilde{k}}\rho_{\tilde{k}}}{N_{\tilde{k}} - 1 + \rho_{\tilde{k}}}$$

$$N_{\tilde{k}} + \rho_{\tilde{k}} N_{\tilde{k}+1} - \left(N_{\tilde{k}} - 1 + \frac{(N_{\tilde{k}+1} + 1)N_{\tilde{k}}\rho_{\tilde{k}}}{N_{\tilde{k}} - 1 + \rho_{\tilde{k}}}\right) = \rho_{\tilde{k}} N_{\tilde{k}+1} + 1 - \frac{(N_{\tilde{k}+1} + 1)N_{\tilde{k}}\rho_{\tilde{k}}}{N_{\tilde{k}} - 1 + \rho_{\tilde{k}}} =$$

after a bit of algebra

$$\frac{(N_{\tilde{k}} - 1 - \rho_{\tilde{k}} N_{\tilde{k}+1})(1 - \rho_{\tilde{k}})}{N_{\tilde{k}} - 1 + \rho_{\tilde{k}}}$$

which is clearly positive/negative in the cases it should be, establishing the result.

II. Backing out pass-through rates and slopes

Note that $m_k^\star = \mu_k(c_k + N_k m_k^\star)$ while $m_{k+1}^\star = \frac{\mu_k(c_k+N_k m_k^\star)}{\rho_k(c_k+N_k m_k^\star)}$. Thus $\frac{m_{k+1}^\star}{m_k^\star}$ reveals ρ_k locally. Thus observing relative mark-ups of firms reveals the pass-through rate at all levels but K, as claimed in the text. Similarly note that from $\rho_k, \rho_{k+1}, N_k, N_{k+1}$ and $m_{k+1}^\star$ we can recover ρ_k', as claimed in the text. By Lemma 2

$$\overline{m}^\star \equiv \overline{m}_{k+1}^\star + \overline{m}_k^\star = \left(\frac{N_{k+1}}{\rho_k} + N_k\right)\mu_k$$

therefore by implicit differentiation

$$\frac{d\overline{m}^\star}{dc_{k+1}} = \rho_{k+1}\rho_k - 1 = \eta_k(\rho_{k+1}\rho_k - 1) + \eta_k$$

where

$$\eta_k \equiv \frac{N_{k+1}\rho_k'\mu_k}{\rho_k^2} + \left(\frac{\rho_k - 1}{N_k\rho_k}\right)\left(\frac{N_{k+1}}{\rho_k} + N_k\right)$$

Thus

$$\eta_k = \frac{\rho_{k+1}\rho_k - 1}{\rho_{k+1}\rho_k}$$

Note that by Lemma 2 $m_{k+1}^\star = \frac{\mu_k}{\rho_k}$. Thus, a bit of algebra shows that

$$\rho_k' = \frac{\rho_{k+1}\rho_k - 1}{N_{k+1}m_{k+1}^\star\rho_{k+1}} - \left(\frac{\rho_k - 1}{m_{k+1}^\star N_k N_{k+1}}\right)\left(\frac{N_{k+1}}{\rho_k} + N_k\right)$$

Thus, as claimed, we can recover the slope of ρ_k if we observe the effect of a first order cost shock on prices at level $k+1$.

III. Identification under quantity competition

While all my results apply equally to quantity competition, to identify the relevant quantity pass-through rate is slightly more complex because we typically do not observe shocks to exogenous quantities. However a simple conversion between cost and exogenous quantity shocks solves this problem. By Lemma 2, equilibrium quantity supplied to the market in any GCS quantity competition model can always be written as

$$q^\star - \tilde{q} = f(q^\star)\frac{P(q^\star) - c}{P'(q^\star)}$$

Similarly, by the reasoning above in this appendix, for any k the total quantity supplied by all firms acting at or before that state $q_k^\star$ (including the exogenous quantity $\tilde{q}$) is

$$q_k^\star - \tilde{q} = g_k(q_k^\star)\frac{P_k(q_k^\star) - c}{P_k'(q_k^\star)}$$

where P_k is defined analogously to Q_k. Thus for any k

$$\frac{dq_k^\star}{d\tilde{q}} = -\frac{P_k(q_k^\star) - c}{q_k^\star - \tilde{q}}\frac{dq_k^\star}{dc}$$

If I assume $\tilde{q} = 0$ and let the equilibrium mark-up (of all firms) be $m^\star$ then this simplifies to

$$\frac{dq_k^\star}{d\tilde{q}} = -\frac{m^\star}{q_k^\star}\frac{dq_k^\star}{dc}$$

Thus from cost shocks, quantities and mark-ups I can recover the effect that a shock to exogenous quantity would have on equilibrium quantities. The quantity pass-through rate from q_{k+1} to q_k is then just

$$\rho_k^Q \equiv \frac{\frac{dq_k^\star}{d\tilde{q}}}{\frac{dq_{k+1}^\star}{d\tilde{q}}} = \frac{q_{k+1}^\star}{q_k^\star}\frac{\frac{dq_k^\star}{dc}}{\frac{dq_{k+1}^\star}{dc}}$$

which is trivial to compute if we can observe the first-order effect of a cost shock on quantities at each level. Thus identification under quantity competition requires slightly different formulae, but the basic approach is the same. The same basic techniques can be used to measure the slope of quantity pass-through (alla Section II above) or to recover pass-through rates, as above, at all but the highest level simply by comparing equilibrium quantity levels.

References

Seade, Jesus, "The Stability of Cournot Revisited," *Journal of Economic Theory*, 1980, *23* (1), 15–27.