Constant Pass-through Demand Systems

An appendix to *Pass-through as an Economic Tool*

E. Glen Weyl

October 2009

I. Constant Pass-through Demand Systems

First I show that if Lop and Lxp are constant in all firms' prices then all pass-through rates are and demand takes the form stated in the text. By appendix *Multiple Products* Lop and Lxp are just the entries of the matrix $(I - \nabla\mu[\mathbf{p}])^{-1}$. If the entries of this matrix are constant in $\mathbf{p}$ then clearly so are the entries of $\nabla\mu(\mathbf{p})$. Therefore $\mu(\mathbf{p})$ must be an affine function of $\mathbf{p}$. That is $\mu^i(\mathbf{p}) = \alpha_i + \sum_j a_{ji} p_j$. Thus

$$-\frac{1}{(\log[Q^i])'} = \mu^i(\mathbf{p}) = \alpha_i + \sum_j a_{ji} p_j$$

or

$$\left(\log[Q^i]\right)' = -\frac{1}{\alpha_i + \sum_j a_{ji} p_j}$$

Integrating this differential equation with respect to p_i yields

$$\log(Q^i) = k^i(\mathbf{p}_{-i}) - \frac{\log(\alpha_i + \sum_j a_{ji} p_j)}{a_{ii}}$$

$$Q^i(\mathbf{p}) = e^{k^i(\mathbf{p}_{-i})} \left(\alpha_i + \sum_j a_{ji} p_j\right)^{-\frac{1}{a_{ii}}}$$

for some function k^i. Letting $a_{ii} \equiv \frac{\rho_i}{\rho_i - 1}$, $f^i(\mathbf{p}_{-i}) \equiv \left(\frac{\rho_i}{[1-\rho_i]^2}\right)^{\frac{\rho_i}{1-\rho_i}} e^{k^i(\mathbf{p}_{-i})}$, $\tilde{p}_i \equiv \frac{\alpha_i(1-\rho_i)}{\rho_i}$ and $\beta_{ji} \equiv \frac{a_{ji}(1-\rho_i)}{\rho_i}$ yields the form in the text for CoPaDS.

It should be immediately clearly that oligopoly pricing is linear, as μ is linear and oligopoly pricing is simply the solution to $\mathbf{p} - \mathbf{c} = \mu(\mathbf{p})$. In particular the solution takes the form

$$\mathbf{p}^\star = \mathbf{c} + \alpha + \mathbf{a}^\top \mathbf{p}^\star$$

where $\mathbf{a}$ is the matrix with i, jth element a_{ij}. Therefore the matrix $\mathbf{K}$ discussed in the paper is simply $\left(\mathbf{I} - \mathbf{a}^\top\right)^{-1}$. From my discussion in appendix *Multiple Products*, stability simply requires that $\mathbf{a}^\top - \mathbf{I}$ satisfy some condition, such as being Hicksian matrix or D-stable. So long as $\mathbf{K}$ is non-singular, there is a unique solution equilibrium given by $\mathbf{p}^\star = \mathbf{K}(\mathbf{c} + \alpha)$.

II. Horizontal Constant Pass-through Demand System

Here I show that post-merger equilibria can be identified by the solution to a polynomial equation. Post-merger first-order conditions remain unchanged for non-merging firms. For the merging firms they become

$$p_i - c_i = \mu_i - \frac{Q_i^j}{Q_i^i}(p_j - c_j)$$

where i is one merging product and j is the other. Note that by construction Q_i^j and Q_i^i are polynomial if $\frac{\rho_i}{1-\rho_i}$ is rational and all other first-order conditions are linear. Thus post-merger equilibrium is a polynomial equation in N variables assuming $\frac{\rho_i}{1-\rho_i}$ is rational. However if $\frac{\rho_i}{1-\rho_i}$ is irrational I can approximate it arbitrarily well with a rational number. This polynomial will be of low order when $\frac{\rho_i}{1-\rho_i}$ can be expressed in lowest terms with integer numerator and denominator that are small. Whenever $\log\left(\left|\frac{\rho_i}{1-\rho_i}\right|\right)$ is small in absolute value ($\frac{\rho_i}{1-\rho_i}$ is neither small nor large in absolute value) we can approximate $\frac{\rho_i}{1-\rho_i}$ closely with a rational number with small numerator and denominator. However, when ρ_i is near 0 or 1 the polynomial will have to be of high-order to provide a good approximation and standard, non-analytic numerical methods will be needed. This more pessimistic case is demonstrated below by the exponential example I give.

III. Price Effects of Strictly Anticompetitive Mergers

Our example is something of a worst-case scenario for HCoPaDS. It demonstrates its largest weaknesses, but also how these may not be very germane in applications. First, consider the profit function of the merged firm

$$\pi(p_1, p_2) = (p_1 - .81)e^{.2p_2 - p_1} + (p_2 - 1)e^{.2p_1 - p_2}$$

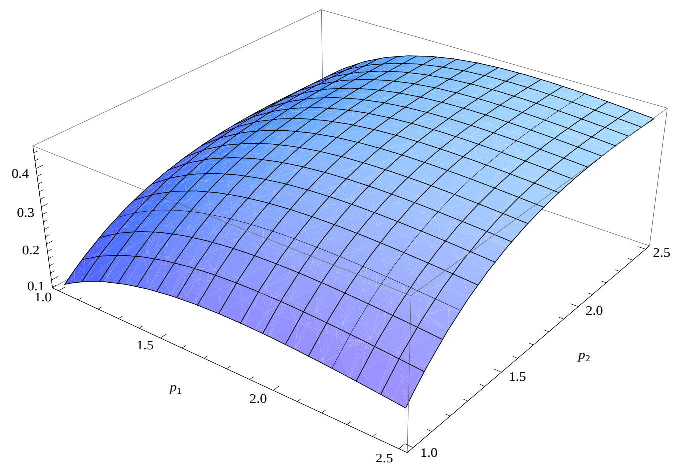

Figure 6: Profit function is concave over $[1, 2.5]^2$

This is quite an implausible profit function and would imply very strange optimal pricing strategy. By setting either p_1 or p_2 to infinity the firm can earn infinite profits on the other good. However, in the local area around the non-cooperative price the profit function looks quite reasonable as shown in Figure 6. It is easily shown, but I omit for brevity a demonstration, that over the range pictured there, $[1, 2.5]^2$, the profit function is concave and maximized at $p_1^\star = 1.98$ and $p_2^\star = 2.38$.

Proof of Proposition 1. To demonstrate that prices are higher on both goods at the merged optimum it is sufficient to show that:

1. Profits are jointly concave in the two prices.

2. The cross partial of the two prices is positive.

If both of these hold then the fact that there is a local incentive to increase both prices at the old equilibrium imply that both prices are higher at the new equilibrium by classic monotone comparative statics results (Milgrom and Roberts, 1990). I show each of these in turn, beginning with concavity.

The post-merger profit function is $\pi(m_1, m_2) = m_1 Q^1(m_1 + c_1, m_2 + c_2) + m_2 Q^2(m_1 + c_1, m_2 + c_2)$ where m_i is the mark-up on good i. The post-merger Hessian is formed of

$$\pi_{11} = 2Q_1^1 + Q_{11}^1 m_1 m_2 Q_{11}^2 = 2Q_1^1 + Q_{11}^1 m_1 + m_2 Q_{22}^2 \left(\frac{Q_1^2}{Q_2^2} \right)^2$$

where the second equality follows by horizontality. The expression for π_{22} is symmetric.

$$\pi_{12} = Q_2^1 + Q_1^2 + m_1 Q_{12}^1 + m_2 Q_{12}^2 = Q_2^1 + Q_1^2 + m_1 Q_{11}^1 \left(\frac{Q_2^1}{Q_1^1} \right) + m_2 Q_{22}^2 \left(\frac{Q_1^2}{Q_2^2} \right) \qquad (21)$$

again by horizontality. Note that I only need to consider the range over which m_i are both positive as m_i negative can never be optimal (reducing m_i below zero can only cause more losses on good i and less profits on good j). To show concavity I must show that π_{ii} is negative and that $\pi_{11}\pi_{22} > \pi_{12}^2$. First I treat π_{11}, by symmetry demonstrating for π_{22}:

$$2Q_1^1 + Q_{11}^1 m_1 + m_2 Q_{22}^2 \left(\frac{Q_1^2}{Q_2^2}\right)^2$$

this is clearly negative as $Q_1^1, Q_{22}^2 < 0 < m_2, m_2$ by concavity. Simplifying $\pi_{11}\pi_{22} - \pi_{12}^2$ yields

$$\pi_{11}\pi_{22} - \pi_{12}^2 = \frac{1}{(Q_2^1 Q_1^2)^2}\Bigg(\left[Q_1^2\right]^2 \left[\left(Q_1^2\right)^2 \left(4Q_1^1 Q_2^2 - \left[Q_1^2 + Q_2^1\right]^2\right) + 2\left(Q_1^1\left[Q_2^1\right]^2 + Q_1^2\left[Q_2^2 Q_1^2 - Q_2^1\left(Q_2^1 + Q_1^2\right)\right]\right) Q_{11}^1 m_1\right] \tag{22}$$

$$+2\left[Q_1^2\right]^2 \left[Q_1^1\left(Q_2^1\right)^2 + Q_1^2\left(Q_2^2 Q_1^2 - Q_2^1\left[Q_2^1 + Q_1^2\right]\right)\right] Q_{22}^2 m_2\Bigg)$$

Because $\left(Q_2^1 Q_1^2\right)^2$ is clearly positive, it is sufficient to show that each of the terms within the outer parentheses are positive The first term has the same sign as

$$\left(Q_1^2\right)^2 \left(4Q_1^1 Q_2^2 - \left[Q_1^2 + Q_2^1\right]^2\right) + 2\left(Q_1^1\left[Q_2^1\right]^2 + Q_1^2\left[Q_2^2 Q_1^2 - Q_2^1\left(Q_2^1 + Q_1^2\right)\right]\right) Q_{11}^1 m_1$$

The first term of this is clearly positive as less-than-perfect diversion and decreasing consumption when both prices increase imply that $Q_i^i > Q_i^j, Q_j^i$. The second term is positive as $m_1 > 0 > Q_{11}^1$ and $Q_i^i < 0 < Q_j^i$. The second term of expression (22) is positive by the same reasoning, establishing concavity.

Now I turn to complementarity. I want to show that $\pi_{12} > 0$. By equation (21) I need to show that

$$Q_2^1 + Q_1^2 + m_1 Q_{11}^1 \left(\frac{Q_2^1}{Q_1^1}\right) + m_2 Q_{22}^2 \left(\frac{Q_1^2}{Q_2^2}\right) > 0$$

This clearly holds by the same reasoning as above, establishing the proposition.

References

Milgrom, Paul and John Roberts, "Rationalizability, Learning and Equilibrium in Games with Strategic Complementarities," *Econometrica*, 1990, *58* (6), 1255–1277.