Stock-Flow Matching*

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Abstract

This paper develops and quantifies the implications of the stock-flow matching model for unemployment, job vacancies, and worker flows. Workers and jobs are heterogeneous, so most worker-job pairs cannot profitably match, leading to the coexistence of unemployed workers and job vacancies. Productivity shocks cause fluctuations in the number of active jobs, which in turn cause fluctuations in labor market outcomes. We derive exact expressions for employment and worker transition rates in a finite economy and analyze their limiting behavior in a large economy. A calibrated version of the model is consistent with the co-movement of labor market variables observed in U.S. data and can explain about one third of their volatility.

1. Introduction

This paper develops and quantifies the implications of the stock-flow matching model (Taylor, 1995; Coles and Muthoo, 1998; Coles and Smith, 1998) for labor market outcomes. Workers and jobs are heterogeneous, so most worker-job pairs cannot profitably match, leading to the coexistence of unemployed workers and job vacancies. Productivity shocks affect the number of jobs that firms create, which in turn causes fluctuations in unemployment, job vacancies, and worker flows. We derive exact expressions for these variables in a finite economy and quantitatively analyze their limiting behavior in a large economy.

Suppose there are L workers and M jobs in the economy at some point in time. Furthermore assume that E workers are employed in a productive match and there is no possibility of productively matching any of the $U = L-E$ unemployed workers with any of the $V = M-E$ vacant jobs. If an idiosyncratic shock causes a filled job to exit the labor market, its worker looks at the stock of vacant jobs to see if she can productively match with one. If not, she joins the stock of unemployed workers. Similarly, when a firm finds it profitable to create a new job, it examines the stock of unemployed workers to see if one is suitable. If not, the job becomes vacant. Thus the inflow of newly unemployed workers matches with the stock of available jobs and symmetrically the stock of unemployed workers matches with the inflow of new jobs, the eponymous stock-flow matching.

The number of employed workers is a random variable, even conditional on the number of workers and jobs. We derive an exact formula for the distribution of the number of employed workers as a function of the current number of workers and jobs and the probability that any worker-job match is productive. We also derive an exact formula for the probability that the entry of a new job leads to an unemployed worker finding a job and for the probability that the exit of a job leads to an employed worker becoming unemployed.

We then consider a limiting version of the economy where the expected number of workers who can productively match with a job α remains fixed, but the number of workers goes to infinity. In the large economy, the employment rate is no longer a random variable, but simply depends on the contemporaneous ratio of the number of jobs to workers, m, and the parameter α. Similarly, the probability that a job exiting causes an employed worker to become unemployed and the probability that a job entering causes an unemployed worker to become employed are deterministic functions of m and α.

Finally, we quantitatively examine how the economy responds to aggregate productivity shocks. The behavior of the calibrated model is nearly indistinguishable from the "mismatch" model in Shimer (2006). It replicates two robust features of the U.S. labor market: the negative correlation between unemployment and vacancies at business cycle frequencies (the *Beveridge curve*) and the positive correlation between the rate at which unemployed workers find jobs and the vacancy-unemployment (v-u) ratio (the *reduced-form matching function*).

In the model and in the data, vacancies are slightly more volatile than unemployment and the correlation between the two variables is strongly negative. The model predicts that a ten percent increase in the v-u ratio should be associated with a two percent increase in the job finding rate. In particular, the elasticity of the model-generated reduced-form matching function is virtually constant. Empirically the elasticity is constant but closer to 0.3.

The calibrated model explains more than a quarter of the volatility in the job finding rate, more than a third of the volatility in the v-u ratio, and more than 40 percent of the volatility in the separation rate of employed workers to unemployment in response to small productivity shocks. These numbers are much larger than the corresponding values that Shimer (2005) found in a search and matching model based on Pissarides (1985).

Previous research on stock-flow matching models has focused either on wage setting (Taylor, 1995; Coles and Muthoo, 1998) or on the model's implication that the number of matches depends on both the stock and inflow of unemployed workers and job vacancies (Coles and Smith, 1998; Coles and Petrongolo, 2003; Smith and Kuo, 2006); see also Lagos (2000). They have typically also assumed that all matches last forever, which simplifies the exposition but limits the possibility of using the model to match labor market facts. In particular, these papers have not derived the distribution of employment or the transition rate from employment to unemployment and back in the finite economy, nor the limiting behavior in the large economy; and they have not shown that stock-flow matching is quantitatively consistent with the empirical Beveridge curve and reduced-form matching function.

This paper also contributes to the search (Lucas and Prescott, 1974) and matching (Pissarides, 1985; Mortensen and Pissarides, 1994; Pissarides, 2000) literature and especially to recent attempts to evaluate the matching model's ability to explain the business cycle behavior of unemployment, vacancies, and worker flows (e.g. Shimer, 2005; Hall, 2005). Our approach here abandons two of the key assumptions in the search and matching model. Rather than posit the existence of a stable matching function, we derive a matching process explicitly from the microeconomic heterogeneity. And rather than assume wages are set via Nash bargaining, we assume that a firm is able to drive down the wage of a worker who has no other job opportunities, but must pay a high wage to a worker with another opportunity. This version of the Mortensen (1982) rule ensures that the decentralized equilibrium maximizes the output produced in the economy. Fortuitously, going back to first principles on the matching and wage determination significantly improves the ability of the model to match the cyclical behavior of labor markets.

Finally, this paper is most closely related to Shimer (2006). In that paper, workers and jobs are located in distinct labor markets, corresponding to occupations or geographic locations. Any worker in a labor market can take any job in the labor market, and each labor market clears with wages determined competitively. By assumption, the allocation of workers and jobs to labor markets is random, and so there are unemployed workers in some labor markets and vacancies in others. In the current paper, whether a worker and job can productively match is idiosyncratic, independent across workers and jobs. Although the microeconomic matching structure is different, the quantitative results shown in Table 2 in that paper and Table 3 in this paper are remarkably similar. This suggests to us that these results may be more general than either particular model.

In any case, the frictions analyzed in search models, mismatch models, and stock-flow matching models are complementary. A more comprehensive model would recognize that there are distinct labor markets with poor possibilities of substituting workers across labor markets, as in Shimer (2006); that not every worker can take every job within a labor market, as in this paper; and that locating a suitable job may require some time-consuming search, as in Lucas and Prescott (1974).

The next section describes our model. Section 3 characterizes the equilibrium of the finite economy. Section 4 considers the behavior of the limiting economy with many workers and jobs. Section 5 calibrates the model and evaluates its quantitative performance. Section 6 briefly concludes.

2. Model

2.1. Idiosyncratic Heterogeneity

We study a continuous time, infinite horizon model. At any point in time t, there are a finite number L workers and a large number of firms. Each firm may have at most one open job. While the number of workers is exogenous, the number of jobs $M(t)$ is determined endogenously by firms' job creation decision discussed below.

Every job is described by a time-invariant L-vector of zeros and ones. Element i of the vector tells us whether the job has a productive match with worker i (1) or not (0). When a firm creates a new job, it immediately realizes the value of this vector, with $x \in (0, 1)$ the probability that any particular worker-job pair is unproductive. The realization of each of these random variables is independent across worker-job pairs.

A productive worker-job pair can jointly generate a flow $p(t)$ units of the numeraire homogeneous consumption good if matched at time t, while an unproductive or unmatched pair yields nothing. A single (unemployed) worker produces $z \in (0, p(t))$ units of the same good at home, while a single (vacant) job produces nothing. These stark assumptions give a concrete notion of unemployment and vacancies.

2.2. Preferences, Wage Setting, and Job Creation

Each worker is risk-neutral and infinitely-lived. She may be either unemployed or employed by a job which has a productive match with her. An employed worker's wage can take on one of two values: if she has a productive match with a job that is currently vacant, competition with the potential employer drives the wage up to $p(t)$; otherwise, the worker receives her leisure value z. In other words, wages are determined by Bertrand competition between employers. Many other wage setting rules are conceivable in a stock-flow matching model; see Taylor (1995), Coles and Muthoo (1998), and Coles and Smith (1998) for three examples. We choose this one because it has a desirable efficiency property (see Section 4.4); however, we stress that wage setting is not important for many the mechanics of unemployment, vacancies, and worker flows.

Firms are also risk-neutral and infinitely-lived. Any firm without a job may create one by paying a sunk cost $k > 0$. A job may be either vacant or filled by one worker who has a productive match with it. A job is hit by an idiosyncratic productivity shock with arrival rate l, rendering it permanently unproductive. When this shock hits, the firm closes the job, laying off its employee if it has one. The firm may then create a new job, but each worker's ability to match productively with it is drawn anew. The arrival of the layoff shock is independent across jobs and over time.

2.3. Matching

There are no search frictions in this economy, only the frictions caused by idiosyncratic heterogeneity. Suppose there are $M(t)$ jobs and $E(t)$ employed workers, each in a productive match with one job, at time t. Also assume none of the $U(t) = L - E(t)$ unemployed workers has a productive match with any of the $V(t) = M(t) - E(t)$ vacant jobs. If no jobs enter and no jobs leave, the matching between workers and jobs is unchanged. If a firm creates a new job, it is filled if it has a productive match with at least one unemployed worker; if it has more than one, we assume it hires the worker who has been unemployed the longest.[1] Otherwise the job is vacant. Conversely, suppose a job exits. If it was vacant, the matching between workers and jobs is unchanged. Otherwise, if the displaced worker has at least one productive match among the vacancies, she immediately takes the job that has been open the longest. If she has no productive match, she becomes unemployed. Note that this behavior ensures that at almost every instant, no unemployed worker has a productive match with a

[1] Our assumption that an entering firm hires the worker who has been unemployed the longest has a small effect on our quantitative conclusions; see Section 3.3. The assumption that a newly-unemployed worker takes the job that has been vacant the longest turns out to be irrelevant.

vacant job.

To implement this matching behavior, we require a complete ordering over jobs based on the time that they were created. If two or more jobs were created at the same instant, as may happen after a positive productivity shock, we assume the order of the jobs is determined by a coin flip. We show that the outcome of the coin flip affects the probability that each job is later subsequently filled and the wage that it pays if it is filled; however, these effects offset so the coin flip does not affect the expected profitability of the job.

Finally, we note that, although this matching behavior ensures that unemployed workers and job vacancies never have a productive match, breaking up existing matches and rematching workers and jobs would often reduce unemployment and vacancies. For example, suppose there are two workers and one job. Both workers can productively match with the job, but it is held by worker 1. Now a new job enters and has a productive match with worker 1 but not with worker 2. Under our assumption, worker 1 stays matched with the old job while worker 2 is unemployed and the new job is vacant. Reallocating worker 1 to the new job would free the old job for worker 2. We assume that this cannot happen, perhaps reflecting some unmodeled turnover cost or job-specific human capital.

2.4. Aggregate Shock

We focus on a single type of aggregate shock, fluctuations in aggregate productivity $p(t)$, but our analytical results extend to fluctuations in other parameters. Assume

$$p(t) = p_{y(t)} = \exp y(t) + \big(1 - \exp y(t)\big)\underline{p} \tag{1}$$

where $y(t)$ is a jump variable lying on a finite grid:

$$y \in Y \equiv \{-\nu\Delta, -(\nu-1)\Delta, \ldots, 0, \ldots, (\nu-1)\Delta, \nu\Delta\}.$$

$\Delta > 0$ is the step size and $2\nu + 1 \geq 3$ is the number of grid points.

A shock hits y according to a Poisson process with arrival rate λ. The new value y' is either one grid point above or below y:

$$y' = \begin{cases} y + \Delta \\ y - \Delta \end{cases} \quad \text{with probability} \quad \begin{cases} \frac{1}{2}\left(1 - \frac{y}{\nu\Delta}\right) \\ \frac{1}{2}\left(1 + \frac{y}{\nu\Delta}\right) \end{cases}.$$

The probability that $y' = y + \Delta$ is smaller when y is larger, falling from 1 at $y = -\nu\Delta$ to 0 at $y = \nu\Delta$. This implies y tends to revert to its mean of zero. Indeed, Shimer (2005) shows that one can represent the stochastic process for y as

$$dy = -\gamma y dt + \sigma db,$$

where $\gamma \equiv \lambda/\nu$ measures the speed of mean reversion and $\sigma \equiv \sqrt{\lambda}\Delta$ is the instantaneous standard deviation.[2] To save on notation, let $\mathbb{E}_p X_{p'}$ denote the expected value of an arbitrary state-contingent variable X following the next aggregate shock, conditional on the current state p.

2.5. Decentralized Equilibrium

In a decentralized equilibrium, each firm decides at each date t whether to create a new job as a function of the current values of productivity $p(t)$ and the number of jobs $M(t)$, taking as given the behavior of all other firms. Let $H^t \equiv \{M(\tau), p(\tau)\}_{\tau=-\infty}^{t}$ denote the observable history of the economy up to and including time t.

[2] Suppose one changes the three parameters of the stochastic process, the step size, arrival rate of shocks, and number of steps, from (Δ, λ, ν) to $\left(\Delta\sqrt{\varepsilon}, \frac{\lambda}{\varepsilon}, \frac{\nu}{\varepsilon}\right)$ for any $\varepsilon > 0$. This does not change either γ or σ, but as $\varepsilon \to 0$, y converges to an Ornstein-Uhlenbeck process.

We assume firms do not observe the number of employed workers $E(t)$, but they understand how the entry and exit of jobs affects the probability that their job is filled and the wage that they must pay if it is filled. In particular, a firm recognizes that at any future date t', it will earn a profit $p(t') - z$ if and only if it has an active job, the job filled, and the employee has no matches among the job vacancies. Otherwise the firm's profit is zero. In the next section, we derive a simple formula for the probability that an active job is filled by a worker with no matches among the job vacancies.

The assumption that a firm cannot observe the realized number of employed workers allows us to characterize the equilibrium recursively, first studying how the distribution of employment depends on the number of workers and jobs and the parameter x which governs frictions, and then looking at firms' entry and exit decision. If firms could observe employment, the distribution of employment conditional on the number of jobs would depend on all the parameters of the model. We conjecture that whether employment is observable does not affect the large economy limit, where the employment rate conditional on the number of jobs is deterministic.

3. Characterization

We now provide a complete characterization of the equilibrium. We first construct an algorithm that tells us which worker is matched to which job at any point in time. This implies a form of history-independence: the distribution of employment at time t conditional on the history of productivity and the number of jobs until time t, H^t, in fact depends only on the number of jobs at time t, $M(t)$; and similarly for the probability that that the entry a job causes an unemployed workers to become employed and the probability that the exit of a job causes an employed worker to become unemployed. We then find analytic expression for these distributions. Finally, we use these mechanical relationships to describe firms' decision to create a job.

3.1. A Matching Algorithm

We start by describing an algorithm that computes the matching pattern at any point in time t. To execute this algorithm, we need to know how long since each worker was last displaced from a job and the ordering over jobs based on the time that they were created.

Definition 1. The *matching algorithm* takes a set of L workers and $M(t)$ jobs and iteratively assigns workers to jobs. Let j_a denote the a^{th}-oldest job.

1. Set $a = 1$ and start with all the workers unmatched.

2. If job j_a can productively match with at least one of the workers whom the algorithm has not yet matched, assign it the one who has experienced the longest time since her last displacement. Otherwise job j_a is vacant.

3. Increase a by 1. If $a \leq M(t)$, repeat step 2.

We stress that this is not a description of how matching happens in the decentralized economy, but rather a computational algorithm that allows us at any point in time to compute who matches with whom. It is useful because it provides a simple way of establishing some important characteristics of the decentralized equilibrium.

PROPOSITION 1. At any time t, the assignment of workers to jobs in the decentralized economy is the same as the one given by the matching algorithm.

The proof is in Appendix A.

3.2. History Independence

We are interested in analyzing how a few summary statistics depend on current and past labor market conditions. Let $\tilde{\phi}(E|H^t)$ denote the distribution of employment at time t conditional on history H^t; let $\tilde{\Pi}^{UE}(E^t, H^t)$ denote the probability that the entry of a job at time t leads to an unemployed worker finding a job, conditional on the observable history H^t and the unobservable history of employment $E^t \equiv \{E(\tau)\}_{\tau=-\infty}^{t}$; and let $\tilde{\Pi}_j^{EU}(E^t, H^t)$ denote the probability that the exit job j at time t leads to an employed worker becoming unemployed conditional on the same variables. The following corollary simplifies this task:

Lemma 1. *For any observable history H^t and unobservable history E^t,*

$$\tilde{\phi}(E|H^t) = \phi(E|M(t)),$$
$$\tilde{\Pi}^{UE}(E^t, H^t) = \Pi^{UE}(E(t), M(t)),$$
$$\text{and } \tilde{\Pi}_j^{EU}(E^t, H^t) = \Pi_j^{EU}(E(t), M(t)).$$

Proof of Lemma 1. Consider the distribution of the employment rate $\tilde{\phi}(E|H^t)$. The matching algorithm constructs the realized matching pattern just from knowledge of the current number of jobs, the entry order of workers and jobs, and the ability of each job-worker pair to match. Since each entry order is equally likely and the ability of each job-worker pair to match is a binomial random variable, the probability distribution of the employment rate depends on the current number of jobs alone. The proof of the other results is similar. ■

In the remainder of our analysis, we simplify notation and the state-space by dropping the history dependence of these variables

3.3. Employment Distribution

We now use the matching algorithm to find the probability distribution over the number of employed workers at each moment.

Proposition 2. The probability that there are $E \in \{0, 1, 2, \ldots\}$ employed workers when there are M jobs is

$$\phi(E|M) = x^{(L-E)(M-E)} \prod_{i=0}^{E-1} \frac{(1 - x^{L-i})(1 - x^{M-i})}{1 - x^{i+1}}. \tag{2}$$

The proof is in Appendix A.

Proposition 2 provides a precise characterization of the distribution of employment conditional on the current number of jobs. Kemp (1998) calls ϕ the "absorption distribution" and describes several environments where it may arise, including an unrelated birth-death process. Her characterization of the moment generating function for the absorption distribution is critical for many of our results.

The assumption that a newly unemployed worker takes the oldest available vacancy affects our exact characterization of the employment distribution. As a vacancy ages, we learn that it is unable to match with a growing fraction of the workforce—all those who have experienced an unemployment spell since the job entered—which implies the vacancy is less likely to be able to match in the future. This has a quantitatively small effect on the employment distribution in the cases we have studied. For example, consider an economy with no aggregate shocks, so whenever a job exits, a new one immediately enters. Suppose that as soon as a filled job exits, the newly unemployed worker takes the *youngest* vacancy with which she can match. After that, a new job enters and hires the unemployed worker with the *shortest* unemployment duration. Let there be $L = 1000$ workers and $M = 968$ jobs (except in the instant after a job exits). Also set $x = 0.9806$, so the average job can hire $\alpha = L(1 - x) = 19.4$ workers.[3] If workers take the oldest available job and firms

[3]We argue in Section 5 that an appropriately scaled version of these values is consistent with the average

hire the longest-term unemployed worker, the average unemployment rate is 5.374 percent with a standard deviation of 0.341 percent, as can be confirmed directly from equation (2). Monte Carlo simulations show that the alternative assumption raises the unemployment rate by 0.014 percentage points and has no effect on the standard deviation. This difference is quantitatively irrelevant.

3.4. Worker Flows

We next use the matching algorithm to find the probability that the entry of a job allows an unemployed worker to find a job.

Proposition 3. The probability that the entry of a new job leads to an unemployed worker finding a job when there are already E employed workers and M jobs is

$$\Pi^{UE}(E, M) = 1 - x^{L-E}. \tag{3}$$

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unemployment and vacancy rates in the U.S. economy. Moreover, the results we report in this paragraph are nearly unchanged in an economy that is ten times larger, with $L = 10,000$ workers, $M = 9680$ jobs, and $x = 0.99806$, so $\alpha = 19.4$.

Proof of Proposition 3. The entry of a new job leads to an unemployed worker finding a job if the new job can match with one of the unemployed workers, with probability $1 - x^U$, where $U = L - E$ is the number of unemployed workers. ■

The expression for the probability that the exit of a job causes an employed to become unemployed is almost as simple but somewhat more cumbersome to derive.

Proposition 4. *The probability that the exit of job j causes an employed worker to become unemployed when there are already E employed workers and M jobs is*

$$\Pi_j^{EU}(E, M) = \frac{x^{-E} - 1}{x^{-M} - 1}. \quad (4)$$

The proof is in Appendix A.

We stress that $\Pi_j^{EU}(E, M)$ does not depend on which job exits and hereafter suppress its dependence on j. When a newer job exits, it is more likely to be vacant. When an older job exits, it is more likely that it is filled by a worker who can immediately move to another job. Perhaps surprisingly, these effects offset, so the probability that an employed worker becomes unemployed when a job exits does not change with the age of the job.

$\Pi^{EU}(E, M)$ is also an important determinant of wages, which are critical to the job creation decision that we analyze next. Recall that a job produces nothing if it is vacant and that it pays its worker her productivity if the worker has an employment opportunity at a vacant job. A job earns positive profit only if it is filled by a worker who would become unemployed if the job exits, with probability $\Pi^{EU}(E, M)$. That this probability is independent of the job's age further simplifies our analysis.[4]

One can verify algebraically that $\Pi^{UE}(E-1, M-1)\phi(E-1|M-1) = \Pi^{EU}(E, M)\phi(E|M)$. This is a statement that worker flows balance: the left hand side is the probability that there are $E-1$ employed workers when there are $M-1$ jobs and the entry of the M^{th} job leads to an unemployed worker finding a job. The right hand side is the probability that there are E employed workers when there are M jobs and the exit of one of the jobs leads to an employed worker becoming unemployed. Similarly,

$$\sum_{E=0}^{\infty} \Pi^{EU}(E, M)\phi(E|M) = \sum_{E=0}^{\infty} E(\phi(E|M) - \phi(E|M-1)),$$

so the expected decrease in employment when one of M jobs exits is just the expected difference in employment between an economy with M and $M-1$ jobs.

[4]It also justifies our earlier assumption that when two or more jobs enter at the same instant, the order of the jobs is determined randomly: there is no incentive to create a job slightly earlier than the competition.

3.5. Job Creation Decision

We finally consider the decision of a firm to open a job when current productivity is p and the current number of jobs is M. The decentralized equilibrium is characterized by a sequence of targets M_p^*. If $M < M_p^*$ when productivity is p, firms instantaneously create $M_p^* - M$ jobs. If $M = M_p^*$, any job that ends is immediately replaced. If $M > M_p^*$, no jobs are created. To describe these targets, we write a Hamilton-Jacobi-Bellman equation for the value of a job $\mathcal{J}_p(M)$ as a function of the current state (p, M); the preceding analysis implies that history is not payoff-relevant.

If $M > M_p^*$, no new jobs are created so

$$r\mathcal{J}_p(M) = (p - z)\sum_{E=0}^{\infty} \Pi^{EU}(E, M)\phi(E|M) - l\mathcal{J}_p(M) \\ + l(M-1)\big(\mathcal{J}_p(M-1) - \mathcal{J}_p(M)\big) + \lambda\big(\mathbb{E}_p\mathcal{J}_{p'}(M) - \mathcal{J}_p(M)\big). \quad (5)$$

The first term on the right hand side is the profit from a filled job paying the low wage z, times the probability that a job is filled by a worker with no opportunities among the vacancies. This is simply the expected value of $\Pi^{EU}(E, M)$ defined in equation (4), where the expectation recognizes the randomness of the number of employed workers E given the number of jobs M. The second term is the probability that the job ends, leading to a capital loss of $\mathcal{J}_p(M)$. The third term is the possibility that one of the $M-1$ other jobs end, leading to a capital gain of $\mathcal{J}_p(M-1) - \mathcal{J}_p(M)$. The final term is the possibility of an aggregate shock, which changes productivity from p to p' as described in Section 2.4 and possibly leads to the immediate entry of one or more jobs.

Second, if $M = M_p^*$, job creation and destruction balance so the Hamilton-Jacobi-Bellman equation simplifies slightly:

$$r\mathcal{J}_p(M_p^*) = (p-z)\sum_{E=0}^{\infty}\Pi^{EU}(E, M_p^*)\phi(E|M_p^*) - l\mathcal{J}_p(M_p^*) + \lambda\big(\mathbb{E}_p\mathcal{J}_{p'}(M_p^*) - \mathcal{J}_p(M_p^*)\big). \quad (6)$$

Moreover,

$$\mathcal{J}_p(M_p^*) \geq k > \mathcal{J}_p(M_p^* + 1), \quad (7)$$

so creating the $M_p^{*\text{th}}$ job is profitable but creating the $(M_p^*+1)^{\text{st}}$ is not. Finally, if $M < M_p^*$, entry immediately drives M up to M_p^* so $\mathcal{J}_p(M) = \mathcal{J}_p(M_p^*)$ as well.

A decentralized equilibrium is characterized by Bellman values $\mathcal{J}_p(M)$ and targets M_p^* that satisfy equations (5)–(7).

4. Large Economy

Although it is possible to work in an economy with a finite number of workers and jobs, it is computationally cumbersome. We show in this section that the finite economy has a relatively simple limit as it grows large. Moreover, we find numerically that convergence to that limit is rapid. Our approach parallels the previous section: we first describe the mechanics of employment and worker flows conditional on the ratio of jobs to workers and then turn to firms' determination of the number of jobs.

Let $m = M/L$ denote the number of jobs per worker at some point in time and $\alpha \equiv L(1-x)$ denote the expected number of workers with whom a job can productively match, an inverse measure of frictions in the economy. We focus on the limiting behavior of the economy as L converges to infinity holding α fixed, for arbitrary values of the endogenous variable m.

4.1. Employment

Proposition 2 provides an exact expression for the probability that there are E employed workers when there are L workers and M jobs. This distribution has a simple limit in the large economy.

PROPOSITION 5. Fix $\alpha = L(1-x)$. For given $m = M/L$, consider the limit as the number of workers L converges to infinity. The fraction of workers who are employed, E/L, converges in mean square to

$$e(m) = 1 + m - \frac{1}{\alpha} \log\big(\exp\alpha + \exp(\alpha m) - 1\big). \tag{8}$$

The proof is in Appendix A.

Manipulation of equation (8) gives the unemployment and vacancy rates as well:

$$u(m) = 1 - e(m) = \frac{1}{\alpha} \log\big(\exp\alpha + \exp(\alpha m) - 1\big) - m \tag{9}$$

$$v(m) = 1 - e(m)/m = \frac{1}{m}\left(\frac{1}{\alpha} \log\big(\exp\alpha + \exp(\alpha m) - 1\big) - 1\right) \tag{10}$$

This implicitly defines the unemployment rate as decreasing in m and the vacancy rate as increasing in m, and hence the vacancy rate as a decreasing function of the unemployment rate for any α, a theoretical Beveridge curve.

The first two rows in Table 1 show the rapid convergence of the unemployment rate and vacancy rate when the job-worker ratio is fixed at 0.968 and there are on average $\alpha = 19.4$

suitable workers per job. We choose these values because they imply unemployment and vacancy rates of 5.417 and 2.291 percent, respectively, in the limiting economy, close to the recent average values in the U.S. economy.[5]

To obtain some intuition for Proposition 5, we can work with a version of the matching algorithm in an economy with a continuum of agents. Order the workers $i \in [0, 1]$ according to the amount of time since they last lost a job, so worker 1 just lost her job. Similarly order the jobs according to the amount of time since they entered, with job 0 the oldest. Then match jobs to workers sequentially, giving job 0 the opportunity to match first. Since there are $1 \equiv u(0)$ workers available, job 0 has a match with probability $1 - \exp(-\alpha u(0))$. In this event, it hires the lowest-named worker. Proceeding sequentially, when job m has the opportunity to match, there are $u(m)$ available workers and so she has a match with probability $1 - \exp(-\alpha u(m))$. With an abuse of the law of large numbers, this suggests $u'(m) = -1 + \exp(-\alpha u(m))$. The solution to this differential equation gives equation (9). Although this ignores all the microeconomic randomness in who matches with whom, it obtains the correct solution.

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[5] We discuss the unemployment and vacancy data in Section 5.

4.2. Worker Flows

We also obtain simple limits for the probability that the entry of a job leads to an unemployed worker finding work and that the exit of a job leads to an employed worker losing her job.

Proposition 6. Fix $\alpha = L(1-x)$. For given $m = M/L$, consider the limit as the number of workers L converges to infinity. The probability that the entry of a new job leads to an unemployed worker finding a job and the probability that the exit of an old job leads to an employed worker becoming unemployed both converge in mean square to

$$\pi(m) = \frac{\exp\alpha - 1}{\exp\alpha + \exp(\alpha m) - 1}. \tag{11}$$

The proof is in Appendix A. The last two rows of Table 1 shows the rapid convergence of Π^{EU} and Π^{UE} to π.

Note that in a large economy, the probability that an entrant hires a worker equals the increase in employment rate from the entry of a single job, $\pi(m) = e'(m)$; we can confirm this equality directly by differentiating equation (8). Symmetrically, this must also equal the probability that a job exiting leads to a worker becoming unemployed.

4.3. Job Creation Decision

Firms' job creation decision is slightly simpler in a large economy. The equilibrium is characterized by a sequence of targets m_p^*. If $m(t) < m_p^*$ when productivity is p, firms instantaneously create enough jobs to raise the job-worker ratio to m_p^*. If $m(t) = m_p^*$, gross job creation and destruction are equal. If $m(t) > m_p^*$, no jobs are created.

To describe these targets, we write a Hamilton-Jacobi-Bellman equation for the value of a job. First, if $m > m_p^*$, no new jobs are created so

$$rJ_p(m) = (p - z)\pi(m) - lJ_p(m) - lmJ_p'(m) + \lambda\big(\mathbb{E}_p J_{p'}(m) - J_p(m)\big). \tag{12}$$

Similar to equation (5), the first term describes the profit from employing a worker with no other job opportunities, the second gives the risk of the job ending, the third gives the capital gain as other jobs end, and the fourth gives the capital gain following a shock. Second, if $m = m_p^*$, job creation and destruction balance and the value of a job exactly equals the cost:

$$rk = (p - z)\pi(m_p^*) - lk + \lambda\big(\mathbb{E}_p J_{p'}(m_p^*) - k\big). \tag{13}$$

Finally, if $m < m_p^*$, entry immediately drives m up to m_p^* so $J_p(m) = k$ as well.

We now provide a constructive existence and uniqueness proof. This also provides a computational algorithm for the targets m_p^*.

Proposition 7. There is a unique equilibrium. In it, the targets m_p^* are increasing.

The proof is in Appendix A; it is essentially the same as the proof in Shimer (2006).

Lower current productivity, which presages lower future productivity, reduces the revenue from creating a job. Equilibrium is restored because the reduction in jobs raises the share jobs filled by a worker who cannot match with a vacancy, $\pi(m)$, lowering expected wages and raising the profit from creating a job back to zero.

4.4. Efficiency

Finally we prove that the decentralized equilibrium maximizes the expected present value of net output per worker. This is a version of the Mortensen (1982) rule: firms have the proper incentive to create jobs if they receive the full marginal product of a filled job when the worker would otherwise be unemployed and nothing otherwise.

Proposition 8. Consider a hypothetical social planner who wishes to maximize the expected present value of net output by choosing how many jobs to create at each instant as a function of current productivity. The solution to the planner's problem coincides with the decentralized equilibrium.

The proof is in Appendix A.

5. Quantitative Evaluation

We now calibrate the large economy model to quantify the cyclical behavior of unemployment, vacancies, and worker flows. We first discuss our choice of the parameter α which governs the level of unemployment and vacancies. We then to the choice of the other parameters and simulate a stochastic version of the model economy. Our approach follows Shimer (2006) closely.

5.1. Beveridge Curve

We start by examining the model-generated unemployment and vacancy rates, given by equations (9) and (10). For a given value of α, these equation implicitly defines the vacancy rate as a function of the unemployment rate. We compare this with U.S. data on unemployment and job vacancies. The Bureau of Labor Statistics (BLS) uses the Current Population

Survey (CPS) to measure the unemployment rate each month. The ratio of unemployment to the sum of unemployment and employment is the unemployment rate.

Since December 2000, the BLS has measured job vacancies using the JOLTS. This is the most reliable time series for vacancies in the U.S.. According to the BLS, "A job opening requires that 1) a specific position exists, 2) work could start within 30 days, and 3) the employer is actively recruiting from outside of the establishment to fill the position. Included are full-time, part-time, permanent, temporary, and short-term openings. Active recruiting means that the establishment is engaged in current efforts to fill the opening, such as advertising in newspapers or on the Internet, posting help-wanted signs, accepting applications, or using similar methods."[6] We measure the vacancy rate as the ratio of vacancies to vacancies plus employment. The dots in Figure 1 show the strong negative correlation between unemployment and vacancies over this time period, the empirical Beveridge curve.

From December 2000 to April 2006, the unemployment and vacancy rates averaged 5.4 percent and 2.3 percent, respectively. Inverting equations (9) and (10), this is consistent with $\alpha = 19.4$ and $m = 0.968$. Now hold α fixed and consider how variation in m, implicitly in response to productivity shocks, affects unemployment and vacancies; this is the line in Figure 1. The fit of the model to the data is excellent. The fact that the level of the model-

[6]See BLS news release, July 30, 2002, available at `http://www.bls.gov/jlt/jlt_nr1.pdf`

generated Beveridge curve fits the data reflects the choice of α. But the fact that the slope and curvature of the model-generated Beveridge curve also fits the data comes from the structure of the model.

Figure 1 is virtually indistinguishable from Figure 1 in Shimer (2006). That the results are so similar in the mismatch and stock-flow matching models suggests to us that the Beveridge curve may simply be an aggregation phenomenon.

5.2. Calibration

This model is parameterized by 9 numbers: the average number of matches per job α, the job termination rate l, the discount rate r, the value of leisure z, the cost of creating a job k, and the four parameters of the stochastic process for productivity: the lower bound $\underline{p}$, the number of steps ν, the arrival rate of shocks λ, and the step size Δ. We keep α fixed at 19.4 and calibrate the remaining parameters of the model to match salient facts about the U.S. economy.

The model is in continuous time and so we normalize a time period to represent a quarter. We set the quarterly discount rate to $r = 0.012$ and let the job termination rate be $l = 0.158$. We choose this latter value to ensure a quarterly separation rate to unemployment of 0.105

in the deterministic steady state with $m = 0.968$ jobs per worker, consistent with average value reported in Shimer (2005).

The productivity process in equation (1) is centered around 1, a normalization. We set the value of leisure to $z = 0.4$. As in the search model, this is a critical parameter for the volatility of aggregate productivity (Hagedorn and Manovskii, 2005). The lower bound on productivity is $\underline{p} = z + \frac{(r+l)k}{1-\exp(-\alpha)}$, the lowest value which ensures that, even in the worst possible state, the unemployment rate stays between 0 and 1; see equation (19), which then implies $\pi(m^*_{-\infty}) = 1 - \exp(-\alpha)$ and hence $m^*_{-\infty} = 0$ by equation (11). We let $\nu = 1000$, $\lambda = 86.6$, and $\Delta = 0.00591022$. This implies a mean reversion parameter of $\gamma = 0.0866$ and a standard deviation of $\sigma = 0.055$ for the latent variable y. We choose these values to match the standard deviation and autocorrelation of detrended productivity in U.S. data. If we change ν, λ, and Δ without altering γ and σ, the results are scarcely affected. Finally, we fix $k = 2.29137$, which implies a 5.4 percent unemployment rate in the deterministic steady state; this matches the mean unemployment rate during the post-war period.

To characterize the equilibrium, we first compute the targets m^*_p for each of the $2\nu + 1$ states following the procedure in the proof of Proposition 7. We then choose an initial value for $p(0)$ and $m(0)$ and select the timing of the first shock t, an exponentially-distributed random variable with mean $1/\lambda$. We compute the number of unemployed workers who finds jobs and the number of employed workers who lose jobs during the interval $[0, t]$. These are slightly complicated because if $m(0) > m^*_{p(0)}$, there is a time interval when no new jobs are created. Similarly, if $m(0) < m^*_{p(0)}$, $m^*_{p(0)} - m(0)$ jobs immediately enter and $u(m(0)) - u(m^*_{p(0)})$ workers find work. We next compute the number of jobs at time t: if $m(0) \leq \exp(lt) m^*_{p(0)}$, $m(t) = m^*_{p(0)}$; otherwise, $m(t) = \exp(-lt)m(0)$ as the number of jobs decays with exits. Finally, we choose the next value of $p(t)$ as described in Section 2.4 and repeat.

At the end of each month (1/3 of a period), we record unemployment, vacancies, cumulative matches and separations, and productivity. We measure the job finding rate f for unemployed workers as the ratio of the number of matches during a month to the number of unemployed workers at the start of the month; if the number of jobs were constant at m during the month, this would equal $lm\pi(m)/u(m)$. We similarly measure the separation rate s as the number of workers who separate to unemployment divided by the number of employed workers at the start of the month; if the number of jobs were constant at m during the month, this would equal $lm\pi(m)/e(m)$. We throw away the first 25,000 years of data to remove the effect of initial conditions. Every subsequent 53 years of model-generated data gives one sample. We take quarterly averages of monthly data and express all variables as log deviation from an HP filter with parameter 10^5, the same low frequency filter that we use on U.S. data. We create 20,000 samples and report model moments and the cross-sample standard deviation of those moments. We compare these results with the U.S. data reported in Shimer (2005, Table 1) and repeated here in Table 2 for convenience.

5.3. Results

Table 3 summarizes the model generated data. The last column shows the driving force, labor productivity. By construction, we match the standard deviation and quarterly autocorrelation in U.S. data. The remaining numbers are driven by the structure of the model.

The first two columns show unemployment and vacancies. Both of these variables only depend on the contemporaneous number of jobs. Thus the model generates a nearly-perfect negative correlation between them, stronger than the empirical correlation of -0.89. The model also explains 38 percent of the observed volatility in vacancies and 31 percent of the observed volatility in unemployment. The theoretical autocorrelations of the two variables are about equal, consistent with the empirical evidence.

The third column shows the v-u ratio, which Shimer (2005) identifies as a key cyclical variable. The model generates 35 percent of its observed volatility. Mortensen and Nagypal (2005) argue that productivity shocks in fact should not explain all of the observed fluctuations in the v-u ratio since the empirical correlation between productivity and the v-u ratio is only 0.4; by their metric, the stock-flow matching model explains nearly all of the productivity-induced fluctuations in the v-u ratio.

The fourth column shows that the model produces 26 percent of the observed volatility in the job finding rate; however, the model fails to generate a sufficiently strong autocorrelation in this variable. The empirical autocorrelation is 0.91, while the theoretical correlation is significantly lower at 0.74. This low autocorrelation is intrinsic to the structure of the model: the job finding probability fluctuates with the inflow rate of new jobs, i.e. in response to *changes* in the number of jobs. In contrast, vacancies and unemployment depend on the stock of jobs. This leads to a correlation between the job finding probability and both the level and change in the v-u ratio. Coles and Petrongolo (2003) argue that this offers a way to test the stock-flow matching model; however, in U.S. data the correlation in levels is remarkably strong. One possible way to reconcile model and data would be to make the marginal cost of job creation increasing in gross job creation; this should dampen the sharp transitory fluctuations in the job finding probability.

Despite this, the model generates a “reduced-form matching function”—a relationship between the job finding probability and the v-u ratio—that is similar to the one in U.S. data. Empirically, a one percent increase in the v-u ratio is associated with a 0.28 percent increase in the job finding probability. The corresponding theoretical elasticity is about 0.21. Moreover, one can test the constant elasticity assumption both in the theory and the data by regressing the log job finding probability on the log v-u ratio and its square. The quadratic term is insignificant at conventional confidence levels in the data and significant only three times in 20,000 simulations of the model.

The fifth column shows that the model generates 41 percent of the observed volatility in the separation rate into unemployment even though there are no fluctuations in the job termination rate l. The flip side of this is that the model produces a strongly procyclical job-to-job transition rate, consistent with the facts reported in Fallick and Fleischman (2004).[7]

Because the model has only one shock, most of the correlations are close to one in absolute value. Moreover, a one shock model probably should not be able to explain all the volatility in vacancies and unemployment; there must be other shocks in the data, e.g. to the cost of investment goods k (Fisher, 2006). Hall (2005), Mortensen and Nagypal (2005), and Rudanko (2006) propose evaluating one shock models by examining the standard deviation of the projection of the detrended v-u ratio on detrended productivity. By this metric, the projection in the data is 0.151 and in the model it is only slightly smaller, 0.134. Similarly,

[7] We do not report the job-to-job transition rate here because data limitations allow us to construct the series only after 1994.

the target for the separation rate should be just 0.039, compared to 0.031 in the model. By these metrics, the stock-flow matching model explains almost all of the volatility in these variables.

6. Conclusion

This paper describes the equilibrium of a stock-flow matching model, where frictions arise because only a few worker-job matches are productive. We derive explicit expressions for the distribution of the unemployment rate, for the probability that a job entering the labor market causes an unemployed worker to find a job, and for the probability that a job exiting the labor market causes an employed worker to become unemployed. These have a simple limit in a large economy, which we use to quantify the implications of the stock-flow matching model for cyclical fluctuations in unemployment, vacancies, and worker flows. The quantitative results are similar to those in Shimer (2006), which suggests that the possibility of a more general approximate aggregation result.

The stock-flow matching model can also be used to examine other labor market issues. For example, suppose match productivity is not simply a binary variable, so workers and jobs must choose a productivity threshold for accepting a partner. In such an environment, labor market policies like unemployment benefits may raise the threshold, reducing the average number of acceptable workers per job, analogous to the parameter α in this paper, and shifting the Beveridge curve away from the origin. This is consistent with evidence in some European countries since 1960 (see, for example, Nickell, Nunziata, Ochel, and Quintini, 2003).

The stock-flow approach also pertains to other other markets. Coles and Muthoo (1998) label the agents in their model "buyers" and "sellers" and discuss the real estate market. Lagos (2003) examines the taxicab market in a related model. Idiosyncratic heterogeneity is likely also important in the marriage market. The stock-flow matching model can also capture other markets where idiosyncratic heterogeneity is less important; the appropriate value of α is simply larger and the equilibrium unmatched rates smaller. Our hope is that our analysis provides a set of tools that will prove useful in studying these problems as well.

A. Omitted Proofs

Proof of Proposition 1. Suppose that for some $a \in \{1, \ldots, M(t)\}$, we have proved that assignment of workers to jobs $\{j_1, \ldots, j_{a-1}\}$ is identical in the decentralized equilibrium and the matching algorithm at time t. This is trivially true for $a = 1$. We prove that the assignment of a worker or a vacancy to job j_a is identical in the decentralized equilibrium and the matching algorithm at time t as well, and so establish the result by induction.

Suppose that in the decentralized equilibrium, job j_a is filled by worker i. By construction, this is a productive match. Now consider any other worker i'. It is impossible that i' has a productive match with j_a and was unemployed for more time than i when i and j_a matched, for then j_a would have matched with i'. The remaining possibilities are that i' does not have a productive match with j_a, or that i' has a productive match but was already matched to some $j_{a'}$ when worker i and job j_a matched, or that i' has a productive match but was unemployed for less time than i when i and j_a matched. We consider each of these possibilities in turn and show that in each case, the matching algorithm does not assign i' to j_a.

First, suppose i' does not have a productive match with j_a. Then the matching algorithm trivially does not assign i' to j_a.

Second, suppose i' has a productive match with j_a but was matched with some job j' when i and j_a matched. This implies that job j' is older than job j_a, for otherwise i' would have matched with j_a in the decentralized equilibrium. If job j' is still open at time t, the induction step implies that the matching algorithm also assigns i' to j' and hence does not assign i' to j_a. If job j' is closed by time t, the time since displacement is shorter for i' than for i, and hence again the algorithm does not assign i' to j_a. Third, suppose i' has a productive match with j_a but was unemployed for less time than i when i and j_a matched. Then at time t, the elapsed time since displacement is again shorter for i' than for i and so the algorithm does not assign i' to j_a.

Finally, the matching algorithm assigns workers to jobs $\{j_1, \ldots, j_{a-1}\}$ exactly as in the decentralized equilibrium, and in particular leaves worker i unmatched. Since job j_a has a productive match with i, the algorithm must assign some worker to job j_a; and since it does not assign any other worker i', it must assign worker i.

Alternatively, suppose job j_a is vacant at time t. Then any worker i' either does not have a productive match with j_a or has a productive match but was already matched to some $j_{a'}$ when job j_a entered. Repeating the same arguments shows that the matching algorithm does not assign i' to j_a, and hence leaves j_a vacant. This completes the induction step and hence the proof. ∎

Proof of Proposition 2. We can trivially prove that $\phi(0|0) = 1$ and $\phi(E|0) = 0$ for any $E > 0$, consistent with equation (2). Moreover, $\phi(0|M) = x^{LM}$ for any M, since there are no employed workers only if all worker-job pairs are unproductive. Again this is consistent with equation (2).

We now proceed by induction. Suppose that we have proved equation (2) for some $M - 1 \geq 0$ and all $E \geq 0$ and want to establish it for M and some $E \geq 1$. Using the matching algorithm and the induction step, the probability that there are $E-1$ matches among the $M-1$ oldest jobs is $\phi(E-1|M-1)$. Conditional on $E-1$ matches among those jobs, the probability the newest job is matched is $1 - x^{L-E+1}$, leaving us with E matches among the M jobs. The other way to attain E matches among the M jobs is if there are E matches among the $M-1$ oldest jobs, with probability $\phi(E|M-1)$, and the newest job is unmatched, with probability x^{L-E}. Putting this together,

$$\phi(E|M) = (1 - x^{L-E+1})\phi(E-1|M-1) + x^{L-E}\phi(E|M-1).$$

Expanding $\phi(E-1|M-1)$ and $\phi(E|M-1)$ using the induction step and simplifying yields equation (2). ■

Proof of Proposition 4. It is trivial that $\Pi_j^{EU}(0, M) = 0$ for all j and M, since there are no employed workers. We can also directly characterize the probability that the exit of the newest job, call it job M, causes a worker to become unemployed; this happens if and only if the job is filled:

$$\Pi_M^{EU}(E, M) = \frac{(1 - x^{L-E+1})\phi(E-1|M-1)}{(1 - x^{L-E+1})\phi(E-1|M-1) + x^{L-E}\phi(E|M-1)} \tag{14}$$

To understand this, partition the configurations with E employed workers and M jobs in two. First, job M is filled, so without the newest job there would be $E-1$ employed workers, with probability $\phi(E-1|M-1)$. Conditional on this, job M is filled with probability $1-x^{L-E+1}$. Second, job M is vacant, so without the newest job there would be E employed workers, with probability $\phi(E|M-1)$. Conditional on this, job M is vacant with probability x^{L-E}. The relatively likelihood of the former configuration gives us the probability the newest job is filled, explaining equation (14). Next, we can verify directly using equation (2) that for all $M \geq 1$ and $1 \leq E \leq \min\{M, L\}$,

$$(1 - x^{L-E+1})\phi(E-1|M-1) = \frac{x^{-E} - 1}{x^{-M} - 1}\phi(E|M)$$
$$\text{and } x^{L-E}\phi(E|M-1) = \frac{x^{-M} - x^{-E}}{x^{-M} - 1}\phi(E|M)$$

Substituting these into equation (14) and simplifying verifies equation (4) for $j = M$.

Now use induction to complete the proof. Suppose we have proved equation (4) for arbitrary $M - 1 \geq 1$ and all $E \in \{0, 1, \ldots, M-1\}$ and $j \in \{0, 1, \ldots, M-1\}$. We extend it to M and all $E \in \{0, 1, \ldots, M\}$ and $j \in \{0, 1, \ldots, M-1\}$ and hence establish the result by induction. We start with the following recursive equation:

$$\Pi_j^{EU}(E, M) = \Pi_j^{EU}(E-1, M-1)\Pi_M^{EU}(E, M) + \Pi_j^{EU}(E, M-1)x\big(1 - \Pi_M^{EU}(E, M)\big). \quad (15)$$

To see this, again partition the configurations with E employed workers and M jobs in two. First, job M is filled, with probability $\Pi_M^{EU}(E, M)$. According to the matching algorithm—which matches workers and jobs identically to the decentralized equilibrium by Proposition 1—the exit of job j when there are E employed workers, M jobs, and job M is filled causes a worker to become employed exactly it when would have with $E - 1$ employed workers and $M-1$ jobs, with probability $\Pi_j^{EU}(E-1,M-1)$. In the second configuration, job M is vacant, with probability $1-\Pi_M^{EU}(E,M)$. Proposition 1 implies that conditional on this, the exit of job j leads to a worker becoming unemployed if it would have led to a worker becoming unemployed with E employed workers and $M-1$ jobs, with probability $\Pi_j^{EU}(E,M-1)$, and the unemployed worker cannot match with job M, with probability x. Now plug the known formulae for $\Pi_j^{EU}(E-1,M-1)$, $\Pi_M^{EU}(E,M)$ and $\Pi_j^{EU}(E,M-1)$ into equation (15) and simplify to establish equation (4). This completes the induction step and the proof. ■

Proof of Proposition 5. We break the proof into three steps to improve readability.

Step 1. Kemp (1998, equation 13) proves that for any L, M, and x, the expected number of employed workers solves

$$\sum_{E=0}^{\infty} E\phi(E|M) = \sum_{E=1}^{\infty} \frac{\prod_{i=0}^{E-1}(1-x^{L-i})(1-x^{M-i})}{1-x^E}. \tag{16}$$

Using $x = 1 - \alpha/L$ and $M = mL$, the employment rate is $e(m) \equiv \lim_{L\to\infty} \sum_{E=1}^{\infty} B_E(L)$, where

$$B_E(L) \equiv \frac{\prod_{i=0}^{E-1} \left(1 - (1-\alpha/L)^{L-i}\right)\left(1 - (1-\alpha/L)^{mL-i}\right)}{L\left(1 - (1-\alpha/L)^E\right)}. \tag{17}$$

Define

$$\bar{B}_E \equiv \lim_{L\to\infty} B_E(L) = \frac{\prod_{i=0}^{E-1} \lim\limits_{L\to\infty} \left(1 - (1-\alpha/L)^{L-i}\right) \lim\limits_{L\to\infty} \left(1 - (1-\alpha/L)^{mL-i}\right)}{\alpha \lim\limits_{L\to\infty} \dfrac{1 - (1-\alpha/L)^E}{1 - (1-\alpha/L)}}$$
$$= \frac{\big((1-\exp(-\alpha))(1-\exp(-\alpha m))\big)^E}{\alpha E},$$

where the last equation uses the fact that for fixed i, $\lim_{L\to\infty}(1-\alpha/L)^{L-i} = \exp(-\alpha)$ and $\lim_{L\to\infty}(1-\alpha/L)^{mL-i} = \exp(-\alpha m)$; and for fixed E, $\lim_{L\to\infty} \frac{1-(1-\alpha/L)^E}{1-(1-\alpha/L)} = E$. This implies

$$\sum_{E=1}^{\infty} \bar{B}_E = -\frac{1}{\alpha} \log\left(1 - (1-\exp(-\alpha))(1-\exp(-\alpha m))\right)$$
$$= 1 + m - \frac{1}{\alpha} \log\left(\exp\alpha + \exp(\alpha m) - 1\right),$$

since $\sum_{E=1}^{\infty} \frac{a^E}{E} = -\log(1-a)$ for any $a < 1$. Equation (8) follows if $\lim_{L\to\infty} \sum_{E=1}^{\infty} B_E(L) = \sum_{E=1}^{\infty} \lim_{L\to\infty} B_E(L)$, that is, if we can switch the order of limits.

Step 2. To prove that we can, note that for all $n \geq 0$,

$$\left| \sum_{E=1}^{\infty} \bar{B}_E - \sum_{E=1}^{\infty} B_E(L) \right| \leq \sum_{E=n+1}^{\infty} \bar{B}_E + \sum_{E=n+1}^{\infty} B_E(L) + \left| \sum_{E=1}^{n} B_E(L) - \sum_{E=1}^{n} \bar{B}_E \right|. \tag{18}$$

We prove that for any $\varepsilon > 0$, there exists an n and an $\bar{L}$ such that each of the terms on the right hand side is smaller than $\varepsilon/3$ for all $L > \bar{L}$.

Start with the first term. Since $\bar{B}_E \geq 0$ and $\sum_{E=1}^{\infty} \bar{B}_E$ is finite, there exists an $\bar{n}_1$ such that $\sum_{E=n+1}^{\infty} \bar{B}_E < \varepsilon/3$ for all $n \geq n_1$.

Next look at the second term. For any $i \geq 0$, $E \geq 1$, and $L > \alpha$, $(1-\alpha/L)^{L-i} \geq (1-\alpha/L)^L$, $(1-\alpha/L)^{mL-i} \geq (1-\alpha/L)^{mL}$, and $(1-\alpha/L)^E \leq 1-\alpha/L$. Then equation (17) implies $B_E(L) \leq z^E/\alpha$, where $z \equiv \big(1-(1-\alpha/L)^L\big)\big(1-(1-\alpha/L)^{mL}\big) \in (0,1)$. It follows that $\sum_{E=n+1}^{\infty} B_E(L) \leq z^{n+1}/\alpha(1-z)$. In particular, for fixed n,

$$\lim_{L\to\infty} \sum_{E=n+1}^{\infty} B_E(L) \leq \lim_{L\to\infty} \frac{z^{n+1}}{\alpha(1-z)} = \frac{\big((1-\exp(-\alpha))(1-\exp(-\alpha m))\big)^{n+1}}{\alpha\big(1-(1-\exp(-\alpha))(1-\exp(-\alpha m))\big)}.$$

The last expression is smaller than $\varepsilon/6$ for all $n \geq n_2$. Fix any $n \geq \max\{n_1, n_2\}$ in the rest of the proof; there exists an $\bar{L}_2$ such that for all $L \geq \bar{L}_2$, the second term in equation (18) is smaller than $\varepsilon/3$.

Now turn to the last term in equation (18). For the given value of n, the last term is smaller than $\varepsilon/3$ for all $L > \bar{L}_3$ since $B_E(L) \to \bar{B}_E$ for all E. Let $\bar{L} = \max\{\bar{L}_2, \bar{L}_3\}$ to complete this step and prove that the expected employment rate is $e(m)$ in a large economy.

Step 3. We finally prove that the variance of E/L converges to zero. Kemp (1998, equation 14) also proves that for any L, M, and x,

$$\sum_{E=0}^{\infty} E(E-1)\phi(E|M) = 2\sum_{E=1}^{\infty}\left(\sum_{i=1}^{E-1}\frac{1}{1-x^i}\right)\left(\frac{\prod_{i=0}^{E-1}(1-x^{L-i})(1-x^{M-i})}{1-x^E}\right).$$

Also note that

$$\lim_{L\to\infty}\sum_{E=0}^{\infty}\left(\frac{E}{L}\right)^2\phi(E|M) = \lim_{L\to\infty}\sum_{E=0}^{\infty}\frac{E(E-1)}{L^2}\phi(E|M),$$

since Step 1 implies $\lim_{L\to\infty}\sum_{E=0}^{\infty}\frac{E}{L^2}\phi(E|M)=0$. Thus

$$\lim_{L\to\infty}\sum_{E=0}^{\infty}\left(\frac{E}{L}\right)^2\phi(E|M)=\lim_{L\to\infty}\frac{2}{\alpha^2}\sum_{E=1}^{\infty}\left(\sum_{i=1}^{E-1}\frac{1-x}{1-x^i}\right)\left(\frac{\prod_{i=0}^{E-1}(1-x^{L-i})(1-x^{mL-i})}{(1-x^E)/(1-x)}\right).$$

Again replacing $x=1-\alpha/L$, switching the order of limits using an argument analogous to Step 2, and taking the same limits as before, we get

$$\begin{aligned}\lim_{L\to\infty}\sum_{E=0}^{\infty}\left(\frac{E}{L}\right)^2\phi(E|M)&=\frac{2}{\alpha^2}\sum_{E=1}^{\infty}\left(\sum_{i=1}^{E-1}\frac{1}{i}\right)\left(\frac{\left((1-\exp(-\alpha))(1-\exp(-\alpha m))\right)^E}{E}\right)\\&=\frac{1}{\alpha^2}\log\left(1-(1-\exp(-\alpha))(1-\exp(-\alpha m))\right)^2=e(m)^2,\end{aligned}$$

where the second line uses $\sum_{E=1}^{\infty}\left(\sum_{i=1}^{E-1}\frac{1}{i}\right)\left(\frac{a^E}{E}\right)=\frac{1}{2}\log(1-a)^2$ for any $a<1$. We have proved that

$$\lim_{L\to\infty}\sum_{E=0}^{\infty}\left(\frac{E}{L}\right)^2\phi(E|M)=\left(\lim_{L\to\infty}\sum_{E=0}^{\infty}\frac{E}{L}\phi(E|M)\right)^2,$$

and hence the limiting variance of E/L is zero.

Finally, convergence of the expected value of E/L to $e(m)$ and of the variance of E/L to zero implies mean square convergence of E/L to $e(m)$. ■

Proof of Proposition 6. Consider a sequence of economies indexed by the number of workers L with L converging to infinity. In an economy with L workers, there are $M = mL$ jobs, $x = 1 - \alpha/L$ probability of a pair being unproductive, and a random number $E(L)$ employed workers, with distribution given by equation (2). Then equation (3) implies

$$\Pi^{UE}(E(L)) = 1 - \left(\left(1 - \frac{\alpha}{L}\right)^L\right)^{1-E(L)/L} \to 1 - \exp(-\alpha(1 - e(m))),$$

since Proposition 5 proves $E(L)/L$ converges to $e(m)$ in mean square. Replace $e(m)$ using equation (8) and simplify to get equation (11).

Similarly, equation (4) implies

$$\Pi^{EU}(E(L), mL) = \frac{\left(\left(1 - \frac{\alpha}{L}\right)^{-L}\right)^{E(L)/L} - 1}{\left(1 - \frac{\alpha}{L}\right)^{-mL} - 1} \to \frac{\exp(\alpha e(m)) - 1}{\exp(\alpha m) - 1}.$$

Algebraic simplification again yields equation (11). ■

Proof of Proposition 7. We start by constructing the unique equilibrium with increasing targets. The last paragraph proves that there is no other equilibrium. Start with the smallest value $p = p_{-\nu\Delta}$ with associated target $m^*_{p_{-\nu\Delta}}$. Following an aggregate shock, productivity increases by one step with certainty and so the target number of job increases to $m^*_{p_{-(\nu-1)\Delta}} > m^*_{p_{-\nu\Delta}}$. If $m = m^*_{p_{-\nu\Delta}}$, the value of a job is k both before and after the shock, $J_{p_{-\nu\Delta}}(m^*_{p_{-\nu\Delta}}) = J_{p_{-(\nu-1)\Delta}}(m^*_{p_{-\nu\Delta}}) = k$. Then evaluating equation (13) at $p = p_{-\nu\Delta}$ and $m = m^*_{p_{-\nu\Delta}}$ gives

$$rk = (p_{-\nu\Delta} - z)\pi(m^*_{p_{-\nu\Delta}}) - lk. \tag{19}$$

This uniquely defines $m^*_{p_{-\nu\Delta}}$ since π is a decreasing function, as can be confirmed directly from equation (11).

We now proceed by induction. Suppose that for some $y > -\nu\Delta$, $y \in Y$, we have shown that the targets $m^*_{p_{y'}}$ are increasing and we have computed $J_{p_{y'}}(m^*_{p_{y-\Delta}})$ for all $y' < y$, $y' \in Y$. For $m \in [m^*_{p_{y-\Delta}}, m^*_{p_y}]$ and $y' < y$, equation (12) implies

$$\begin{aligned} rJ_{p_{y'}}(m) &= (p_{y'} - z)\pi(m) - lJ_{p_{y'}}(m) - lmJ'_{p_{y'}}(m) \\ &\quad + \frac{\lambda}{2}\left(1 + \frac{y'}{\nu\Delta}\right)\left(J_{p_{y'-\Delta}}(m) - J_{p_{y'}}(m)\right) + \frac{\lambda}{2}\left(1 - \frac{y'}{\nu\Delta}\right)\left(J_{p_{y'+\Delta}}(m) - J_{p_{y'}}(m)\right). \end{aligned} \tag{20}$$

In addition, $J_{p_y}(m) = k$ for $m \in [m^*_{p_{y-\Delta}}, m^*_{p_y}]$. This is a system of $\nu + y/\Delta$ differential equations in m with the same number of terminal conditions from the previous induction steps and so we can compute $J_{p_{y'}}(m)$, $m \in [m^*_{p_{y-\Delta}}, m^*_{p_y}]$ for all $y' < y$, $y' \in Y$. The only catch is that we do not yet know $m^*_{p_y}$. To compute it, evaluate equation (13) at p_y and $m = m^*_{p_y}$:

$$rk = (p_y - z)\pi(m^*_{p_y}) - lk + \frac{\lambda}{2}\left(1 + \frac{y}{\nu\Delta}\right)\left(J_{p_{y-\Delta}}(m^*_{p_y}) - k\right), \tag{21}$$

where we use $J_{p_{y+\Delta}}(m^*_{p_y}) = k$ to eliminate the term coming from a positive shock. This uniquely defines $m^*_{p_y}$ since both π and $J_{p_{y-\Delta}}$ are decreasing.

To complete the induction argument, suppose equation (21) defines $m^*_{p_y} \leq m^*_{p_{y-\Delta}}$. Then

$$(p_y - z)\pi(m^*_{p_y}) = (r + l)k < (p_{y-\Delta} - z)\pi(m^*_{p_{y-\Delta}}). \tag{22}$$

The equality uses $J_{p_{y-\Delta}}(m^*_{p_y}) = k$ whenever $m^*_{p_y} \leq m^*_{p_{y-\Delta}}$. The inequality uses equation (20) evaluated at $y' = y - \Delta$ and $m = m^*_{p_{y-\Delta}}$, but drops the capital gain terms; those are all negative-valued since $m^*_{p_y} \leq m^*_{p_{y-\Delta}}$ (by assumption in this paragraph) and $m^*_{p_{y-2\Delta}} < m^*_{p_{y-\Delta}}$ (from the induction assumption). Since $p_y > p_{y-\Delta}$, equation (22) implies $\pi(m^*_{p_y}) < \pi(m^*_{p_{y-\Delta}})$ or equivalently $m^*_{p_y} > m^*_{p_{y-\Delta}}$, a contradiction.

Finally, suppose there were an equilibrium with $m^*_{p_y} \leq m^*_{p_{y-\Delta}}$ for some $y \in Y$. Focus on the largest such y, so either $m^*_{p_y} < m^*_{p_{y+\Delta}}$ or $y = \nu\Delta$, in which case productivity can only decline from p_y. Analogous to the reasoning behind equation (22), we find

$$(p_y - z)\pi(m^*_{p_y}) = (r+l)k \leq (p_{y-\Delta} - z)\pi(m^*_{p_{y-\Delta}}),$$

since a productivity shock when $p = p_y$ and $m = m^*_{p_y}$ does not affect the value of a job (the target goes up), while a productivity shock when $p = p_{y-\Delta}$ and $m = m^*_{p_{y-\Delta}}$ may reduce the value of a job. The inequalities imply $m^*_{p_y} > m^*_{p_{y-\Delta}}$, a contradiction. ■

Proof of Proposition 8. Let $W_p(m)$ denote the expected present value of net output per worker when current productivity is p and the current job-worker ratio is m. We can represent the planner's problem recursively as

$$rW_p(m) = \max_{g \geq 0}\ pe(m) + z(1-e(m)) - kg + W_p'(m)(g - lm) + \lambda \mathbb{E}_p\big(W_{p'}(m) - W_p(m)\big). \quad (23)$$

Here g is the gross increase in the number of jobs per worker and $e(m)$ is the employment rate given in equation (8). The flow value of the planner, $rW_p(m)$, can be divided into three terms. First is current net output, p for each of the $e(m)$ employed workers, z for each of the $1 - e(m)$ unemployed workers, and $-k$ for each job created. Second is the future increases in $W_p(m)$ coming from any net increase in the number of jobs, the difference between gross job creation and deprecation, $g - lm$. Third is the possibility of an aggregate shock, with arrival rate λ, at which point the planner anticipates a capital gain $\mathbb{E}_p\big(W_{p'}(m) - W_p(m)\big)$.

The first order condition for the gross amount of job creation conditional on the current state (p, m) is

$$g_p(m) \geq 0, \quad W_p'(m) \leq k, \quad \text{and} \quad g_p(m)(W_p'(m) - k) = 0. \tag{24}$$

That is, whenever the marginal value of a job is smaller than k, gross job creation is zero and conversely, if some jobs are being created, the marginal value of a job must equal its cost. The envelope condition is

$$rW_p'(m) = (p - z)\pi(m) - lW_p'(m) + W_p''(m)(g_p(m) - sN) + \lambda\mathbb{E}_p\big(W_{p'}'(m) - W_p'(m)\big), \tag{25}$$

where we use $e'(m) = \pi(m)$.

Combining the first order and envelope conditions, we can define the targets m_p^* as follows. First, if $m > m_p^*$, no new jobs are created, $g_p(m) = 0$, so

$$rW_p'(m) = (p - z)\pi(m) - lW_p'(m) - W_p''(m)sN + \lambda\big(\mathbb{E}_pW_{p'}'(m) - W_p'(m)\big). \tag{26}$$

Second, if $m = m_p^*$, $g_p(m) = lm$ and $W_p'(m) = k$, so the envelope condition reduces to

$$rk = (p - z)\pi(m_p^*) - rk + \lambda\Big(\mathbb{E}_p W_{p'}'(m_p^*) - k\Big). \tag{27}$$

Finally, if $m < m_p^*$, entry immediately drives m up to m_p^* so $W_p'(m) = k$. Comparing with equations (12) and (13) establishes that the equivalence of the decentralized and centralized economies, with $J_p(m) = W_p'(m)$ for all p and m. ■

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L	1000	10,000	100,000	∞
U/L	5.374% (0.341)	5.413% (0.108)	5.417% (0.034)	5.417% (0)
V/L	2.245% (0.352)	2.287% (0.112)	2.291% (0.035)	2.291% (0)
Π^{EU}	0.6547 (0.0436)	0.6508 (0.0137)	0.6504 (0.0043)	0.6504 (0)
Π^{UE}	0.6502 (0.0233)	0.6504 (0.0073)	0.6504 (0.0023)	0.6504 (0)

Table 1: Unemployment rate, vacancy rate, probability exit causes unemployment, and probability entry causes employment (standard deviations in parenthesis) with $M = 0.968L$ and $x = 1 - 19.4/L$ and various values of L.

Summary Statistics, quarterly U.S. data, 1951 to 2003

		U	V	V/U	f	s	p
Standard Deviation		0.190	0.202	0.382	0.118	0.075	0.020
Quarterly Autocorrelation		0.936	0.940	0.941	0.908	0.733	0.878
Correlation Matrix	U	1	−0.894	−0.971	−0.949	0.709	−0.408
	V	—	1	0.975	0.897	−0.684	0.364
	V/U	—	—	1	0.948	−0.715	0.396
	f	—	—	—	1	−0.574	0.396
	s	—	—	—	—	1	−0.524
	p	—	—	—	—	—	1

Table 2: Seasonally adjusted unemployment U is constructed by the BLS from the Current Population Survey (CPS). The seasonally adjusted help-wanted advertising index V is constructed by the Conference Board. The job finding rate f and separation rate s are constructed from seasonally adjusted employment, unemployment, and short-term unemployment, all computed by the BLS from the CPS. See Shimer (2005) for details. U, V, f, and s are quarterly averages of monthly series. Average labor productivity p is seasonally adjusted real average output per person in the non-farm business sector, constructed by the Bureau of Labor Statistics (BLS) from the National Income and Product Accounts and the Current Employment Statistics. All variables are reported in logs as deviations from an HP trend with smoothing parameter 10^5.

Model Generated Data (and standard errors)

		U	V	V/U	f	s	p
Standard Deviation		0.058 (0.008)	0.076 (0.010)	0.134 (0.018)	0.033 (0.004)	0.031 (0.004)	0.020 (0.003)
Quarterly Autocorrelation		0.879 (0.030)	0.879 (0.030)	0.879 (0.030)	0.739 (0.060)	0.885 (0.029)	0.878 (0.031)
Correlation Matrix	U	1	−0.999 (0.000)	−1.000 (0.000)	−0.873 (0.029)	0.994 (0.001)	−0.999 (0.001)
	V	—	1	1.000 (0.000)	0.874 (0.030)	−0.992 (0.002)	0.996 (0.002)
	V/U	—	—	1	0.873 (0.030)	−0.993 (0.001)	0.997 (0.001)
	f	—	—	—	1	−0.906 (0.021)	0.870 (0.029)
	s	—	—	—	—	1	−0.995 (0.001)
	p	—	—	—	—	—	1

Table 3: Results from simulations of the benchmark model. See the text for details.

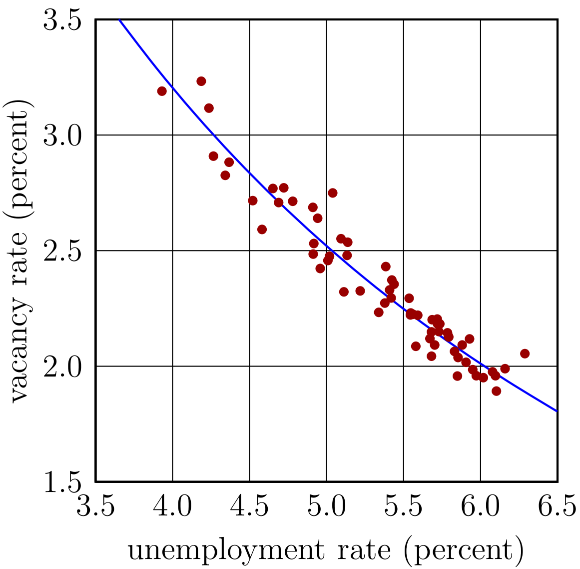

Figure 1: The brown dots show U.S. monthly data from December 2000 to April 2006. The unemployment rate is measured by the BLS from the CPS. The vacancy rate is measured by the BLS from the JOLTS. The line shows the model generated Beveridge curve with $\alpha = 19.4$ and m varying from 0.952 to 0.998.