Time series properties of aggregate output fluctuations*

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The last decade has seen an explosion in research on the unit root component of output, producing contradictory conclusions as to the magnitude of output persistence. This divergence of opinion stems from the examination of different aspects of the first differences of output in order to detect deviations from white noise. This paper explores the time series properties of aggregate U.S. output in order to see whether the data are well described as a random walk with drift. Our testing methodology examines the shape of the spectral distribution function and thereby captures all second moment implications of the null. Our results indicate that there is little statistically significant evidence of mean reversion in output.

1. Introduction

Starting with the work of Nelson and Plosser (1982) on unit roots in macroeconomic time series, a large body of empirical work has attempted to determine the relative importance of permanent versus transitory components in output fluctuations. Much of this concern stems from a belief that only supply side shocks permanently affect aggregate activity. Under this view, the permanent/transitory decomposition is important in revealing the contributions of different structural sources to fluctuations. Recent perspectives in macroeconomic theory have demonstrated that this dichotomy is false, as dynamic coordination failure models with incomplete markets and multiple equilibria can generate unit roots due to either demand or supply side shocks [Durlauf (1989, 1991b, c)]. Further, unit roots occur in these models even if technical change is deterministic. Nevertheless, conjectured sources of aggregate fluctuations such as menu costs

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seem to be intrinsically short-run in nature. Understanding the periodicity of business cycles is germane to establishing the empirical relevance of certain classes of theories.

The empirical literature on output fluctuations has failed to give a definitive answer to the question of the relative significance of permanent output fluctuations. One view, most forcefully articulated by Campbell and Mankiw (1987a, b), has concluded that permanent shocks are an important component of fluctuations. In their analysis of quarterly GNP data, these authors compute a multiplier which represents the marginal change in the expected value of output in the infinite future given a unit innovation today. If \( Y_t \) equals per capita output at \( t \) after removing all deterministic components, then \( \Delta Y_t \) possesses the Wold representation

\[
\Delta Y_t = \sum_{j=0}^{\infty} \varepsilon_{t-j},
\]

where \( \varepsilon_t \) are martingale differences. The Campbell–Mankiw measure of persistence is

\[
E(Y_t | \varepsilon_t = 1) = \sum_{j=0}^{\infty} \varepsilon_j.
\]

Their work, based on real GNP, finds that this limiting change is greater than one.

An alternative view, due primarily to Watson (1986) and Cochrane (1988), is that a substantial component of fluctuations is mean-reverting.¹ Cochrane’s estimate is based on a multiplier which in this case equals the marginal change in the expected value of output given a one unit change in output today. Letting \( \sigma_{YY}(j) \) denote the autocovariance function of \( \Delta Y_t \), simple algebra reveals that

\[
E(Y_t | \Delta Y_t - 1) = \sum_{j=0}^{\infty} E(Y_j | \Delta Y_t - 1) = \sum_{j=0}^{\infty} \frac{\sigma_{YY}(j)}{\sigma_{YY}(0)}.
\]

Cochrane, analyzing log per capita real GNP, finds that this multiplier is substantially less than one.

¹ Watson arrives at a result similar to Cochrane by decomposing the first difference of output into cyclical and trend terms through unobserved components analysis. The unobserved components representation implies that there is long-run mean reversion as the zero frequency value of the trend term is substantially below one. Our argument is that for annual data the trend term can be taken to represent the entire output series. We concentrate on Cochrane’s work in order to focus on the information in the second moments of the output series.
Interestingly, the discrepancies between the Campbell–Mankiw and Cochrane perspectives are not a function of the choice of a particular measure of persistence. As all authors have recognized, one multiplier is a function of the other, since

$$\sum_{j=-z}^{x} \sigma_{\Delta y}(j) = \left( \sum_{j=0}^{x} \gamma_{j} \right)^{2} \sigma_{z}(0).$$

The source of differing estimates comes, instead, from the choice of estimation strategy. Campbell and Mankiw estimate low-order ARMA models, where lag lengths are determined by an information criterion, leading to a parsimonious time domain representation for output changes. This strategy ignores some information in the spectral density if the ARMA specification is misspecified. As demonstrated in Cochrane (1988), misspecification can cause information in the low frequencies to be ignored. Cochrane uses a nonparametric estimation strategy based upon the zero frequency of the spectral density of output changes. High-order autocovariances are allowed to have a significant impact on the estimated long-run behavior of output changes. Cochrane in fact defends his results on the basis that his estimation strategy can reveal long-run mean reversion.

The wide gap between these two views demonstrates the importance of a researcher's prior in determining the outcome of an empirical exercise. Campbell and Mankiw and Cochrane find different results because they are in essence testing different aspects of the same model. If we regard the null model of the two estimation approaches as

$$H_{0}: \quad \text{Output is a random walk with drift},$$

the two research strategies have each rejected the null on the basis of employing tests which are consistent against different alternatives. By examining different aspects of the spectral density of output changes, these authors have come to very different conclusions on the way in which the null is violated.\(^2\)

This paper explores the statistical properties of output changes through a sequence of tests of the model which do not rely upon the choice of a particular alternative. The tests are specifically designed to be consistent against all alternatives to the random walk null. This sort of testing embodies the notion that a researcher has a diffuse prior on the location of an alternative.

Specifically, our testing strategy explores the behavior of the complete spectral distribution function (and hence spectral density) of output changes. Under $H_{0}$, the null hypothesis requires that the spectral distribution function is shaped

\(^2\)Campbell and Mankiw reject the null for quarterly data. For annual data the null is accepted. See Durlauf (1989) for details.
as a diagonal line. All second moment deviations from the null may be formulated geometrically as deviations from this shape. A simple asymptotic theory, developed in Durlauf (1991a), permits the construction of explicit tests of spectral distribution shapes. The spectral shape tests explore both short- and long-run autocorrelations in the data.

At the same time, we analyze the behavior of output changes at different frequencies. By understanding the shape of the spectral distribution function, it will be possible to understand why different studies have arrived at contradictory conclusions as to the nature of deviations from the null.

The application of this testing strategy to aggregate output reveals little evidence against a random walk with drift null for the time periods 1870–1929 and 1947–1989. General spectral goodness-of-fit tests support the view that aggregate output behavior is consistent with a random walk with drift. Further, output behavior in the entire sample 1870–1989 is inconsistent with the random walk specification due to a large concentration of power in cycles of 5 to 16 years rather than due to long-run mean reversion. Over all three sample periods, the data are generally consistent with the view that there is substantial persistence in output fluctuations.

These results should not be read as saying that those authors who have found evidence of mean reversion have been guilty of data mining. We also provide evidence that if one examines the low frequencies of the spectral distribution function in isolation, there is some deficiency of spectral power relative to the null. This is an implication of models of long-run mean reversion. Consequently, our empirical results indicate that rejection of the null requires a strong prior on the possible set of alternatives. If the only admissible alternatives to the random walk are processes with long-run mean reversion, then clearly our tests are relatively unpowful. (Although we shall also see that it is difficult to develop statistically significant rejections of the null for low frequencies as well.) However, if one regards short-run and long-run deviations as equally plausible, then our results support the random walk null. Put differently, if one starts with a prior that aggregate output obeys a random walk with drift, it is difficult to find evidence against this position in favor of a mean-reverting alternative model based on the second moments of output changes.

On the other hand, our analysis represents a contrast to several studies which have argued that unit roots in data are impossible to identify in finite samples. These authors, such as Quah (1988) and Christiano and Eichenbaum (1989), observe that there exist stationary, slowly mean-reverting time series representations which can well approximate any unit root restrictions on finite data samples. We argue, conversely that the data are also consistent with models which are dominated by persistence. Further, we test a point null hypothesis that output is a random walk with drift, rather than a general unit root null, which provides finite sample power against many alternatives. Our work also contrasts with approaches such as Diebold and Rudebusch (1989) who explore
low frequency output behavior in assessing the appropriateness of the unit root hypothesis. The random walk null requires that we explore all spectral properties of output changes.

In terms of economic theory, our results indicate that macroeconomic models which generate persistent output fluctuations are consistent with the history of per capita GNP in the United States. This finding is of particular interest from the perspective of dynamic coordination failure models such as Aghion and Howitt (1989) and Durlauf (1991b) which predict that aggregate output should behave as a random walk with drift.

Section 2 of this paper outlines the basic testing methodology. Section 3 examines the spectral distribution function of output changes. We find that there is little evidence against the null in the pre-Depression and postwar periods. Over the 1870–1987 period the null is rejected. However, the rejection cannot be interpreted as mean reversion. Section 4 discusses point estimates of the spectral density in the context of the issue of long-run mean reversion. The results of the section confirm the spectral distribution function analysis. Section 5 provides summary and conclusions.

2. Spectral implications of the martingale hypothesis

The null hypothesis that output behaves as a random walk with drift places an infinity of restrictions on the data of interest. This occurs because the null restricts all values of the entire autocovariance function except the variance.

$$H_0: \quad \sigma_{yy}(j) = 0 \quad \text{for all} \quad j \neq 0.$$  

Conventional empirical work has concentrated on a subset of these restrictions.

In order to test all the implications of the martingale null, it is useful to map the infinite sequence of autocovariances into the frequency domain. The frequency domain, as we shall see, provides an unambiguous way of distinguishing between long-run and short-run fluctuations. The spectral density of $\Delta Y_t$,

$$f_{yy}(\omega) = \frac{1}{2\pi} \sum_{j=-x}^{x} \sigma_{yy}(j),$$  \hspace{1cm} (4)  

provides this mapping and permits specification of the complete null hypothesis.

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3 We do not consider the issue of nonlinear representations of output. Specifically, we are concerned with the projection of output onto any deterministic functions of time and $L_y(t)$, the space of all square summable linear combinations of $Y_{t-1}, Y_{t-2}, \ldots$. Hamilton (1989) provides some evidence of nonlinearity.
as a statement about the shape of this function. Under the null, the spectral density must be a rectangle.

\[ H_0: \quad f_{1Y}(\omega) = \frac{1}{2\pi} \sigma_{1Y}(0). \]  

(5)

This formulation of the null places a requirement on the spectral density at all frequencies. Zero frequency tests are therefore just one implication of the null.

An equivalent way of formulating the null is in terms of the spectral distribution function, defined for \( \lambda \in [0, \pi] \) as

\[ 2 \int_0^\lambda f_{1Y}(\omega) \, d\omega. \]  

(6)

If the spectral density under the null is a rectangle, then the spectral distribution function is a diagonal line.

\[ H_0: \quad 2 \int_0^\lambda f_{1Y}(\omega) \, d\omega = \frac{1}{\pi} \sigma_{1Y}(0) \lambda. \]  

(7)

The normalized spectral distribution function

\[ F_{1Y}(\lambda) = \frac{2 \int_0^\lambda f_{1Y}(\omega) \, d\omega}{\sigma_{1Y}(0)} \]  

in turn provides a scale free way of assessing \( H_0 \), by seeing whether \( F_{1Y}(\lambda) = (1/\pi) \lambda \).

The behavior of the normalized spectral distribution function provides substantial intuition into the structure of output changes when one recalls that if \( \Delta Y_t \) is a stationary \( L^2 \) process, then the Cramér Representation Theorem [see Ash and Gardner (1975) for a derivation] ensures that the time series can be expressed as

\[ \Delta Y_t = \int_0^{\pi} \cos(\omega t) \, du(\omega) + \int_0^{\pi} \sin(\omega t) \, dv(\omega). \]  

(9)

such that at each fixed \( \omega \), \( du(\omega) \) and \( dv(\omega) \) are zero mean random variables with the properties that i) \( E \, du(\omega)^2 = E \, dv(\omega)^2 \), ii) if \( \omega_1 \neq \omega_2 \), then \( E \, du(\omega_1) \, du(\omega_2) = E \, dv(\omega_1) \, dv(\omega_2) = 0 \), and iii) \( E \, du(\omega_i) \, dv(\omega_j) = 0 \) \( \forall \omega_i, \omega_j \). In other words, output changes can be represented as the sum of a continuum of orthogonal
random variables. Orthogonality means that one can unambiguously measure the contribution of a particular frequency \( \tilde{\omega} \) to the total variance of output changes

\[
\cos^2(\omega) \mathbb{E} u(\omega)^2 + \sin^2(\omega) \mathbb{E} v(\omega)^2 = \mathbb{E} u(\tilde{\omega})^2. \tag{10}
\]

Orthogonality also means that the normalized spectral distribution function gives a precise measure of how different intervals of cycles contribute to the volatility of output movements. The contribution of frequencies \([0, \tilde{\omega}]\) to total variance equals the spectral distribution function. By construction,

\[
\frac{\int_0^\tilde{\omega} \mathbb{E} u(\omega)^2 d\omega}{\sigma_{\Delta Y}(0)} = \frac{2 \int_0^\tilde{\omega} f_{\Delta Y}(\omega) d\omega}{\sigma_{\Delta Y}(0)}. \tag{11}
\]

White noise is a process where all frequencies contribute equally to the variance, i.e., \( \mathbb{E} u(\tilde{\omega})^2 \) is constant. If the normalized spectral distribution function is concentrated below a diagonal line representing white noise, this is evidence that output changes are relatively dominated by quickly reverting components. Cochrane's examination of the zero frequency is an extreme version of examining the low frequencies to see how they contribute to the total variance of output changes.

In order to analyze the properties of the deviations of the empirical spectral distribution function from its theoretical shape under \( H_0 \), define the random function

\[
U_T(t) = \sqrt{2} T^{1/2} \int_0^{\tilde{\omega} t} \left( \frac{I_T(\omega)}{\tilde{\sigma}_{\Delta Y}(0)} - \frac{1}{2\pi} \right) d\omega, \quad t \in [0, 1]. \tag{12}
\]

where \( I_T(\omega) \) is the periodogram of \( \Delta Y \) calculated from \( T \) observations. This formulation computes the deviations of the normalized periodogram-based spectral distribution function from a diagonal. The argument of the function representing the deviations is also normalized so that the sample spectral deviations can be expressed as a random function defined on \([0, 1]\).

Expressing the normalized periodogram \( I_T(\omega)/\tilde{\sigma}_{\Delta Y}(0) \) as

\[
\frac{I_T(\omega)}{\tilde{\sigma}_{\Delta Y}(0)} = \frac{1}{2\pi} \sum_{j=-T+1}^{T-1} \hat{\rho}_{\Delta Y}(j) e^{-ij\omega}, \tag{13}
\]
where the sample autocorrelations $\hat{\rho}_{YY}(\cdot)$ equal

$$
\hat{\rho}_{YY}(i) = \frac{T^{-1} \sum_{j=1+i}^{T} (\Delta Y_j - \overline{\Delta Y})(\Delta Y_{j-i} - \overline{\Delta Y})}{T^{-1} \sum_{j=1}^{T} (\Delta Y_j - \overline{\Delta Y})^2}.
$$

(14)

one can integrate $U_T(t)$ term by term to express the spectral deviations as

$$
U_T(t) = \sqrt{\frac{2}{\pi}} \sum_{j=1}^{T-1} T^{1/2} \hat{\rho}_{YY}(j) \frac{\sin j\pi t}{j}.
$$

(15)

This expression reveals how the asymptotic properties of the spectral deviations depend on the underlying autocorrelations. The $\sqrt{2}$ and $T^{1/2}$ terms provide the appropriate normalization so that the spectral deviations function possesses a nondegenerate limiting distribution.

Developing an asymptotic theory for this random function requires that we place some additional restrictions on $\Delta Y$, which act as part of the null hypothesis.

**Assumption 1. Complete specification of null hypothesis**

The full set of restrictions imposed by $H_0$ is assumed to be:

(i) $E(\Delta Y_j | \mathcal{F}_{j-1}) = \mu$, where $\mathcal{F}_j$ is the $\sigma$-algebra generated by $\Delta Y_k$ for $k \leq j$.

(ii) $E(\Delta Y_j^2) = \sigma^2$.

(iii) $\lim_{T \to \infty} T^{-1} \sum_{j=1}^{T} E(\Delta Y_j^2 | \mathcal{F}_{j-1}) = \sigma^2 > 0$ almost surely.

(iv) There exists a random variable $W$ with $E(W^4) < \infty$ such that $P(|\Delta Y_j| > u) \leq cP(|W| > u)$ for some $0 < c < \infty$ and all $j$, all $u \geq 0$.

(v) $E(\Delta Y_j^2 | \Delta Y_{j-r}, \Delta Y_{j-s}) = \kappa(r, s)$ finite and uniformly bounded $\forall j, r \geq 1, s \geq 1$.

(vi) $\lim_{T \to \infty} T^{-1} \sum_{j=1}^{T} \Delta Y_{j-1} \Delta Y_{j-1} E(\Delta Y_j^2 | \mathcal{F}_{j-1}) = \kappa(r, s)$ almost surely.

(vii) $E(\Delta Y_j^8)$ uniformly bounded $\forall j$.

The first condition generalizes the restrictions on the spectral density to the case of a martingale difference. Conditions (ii) to (vi) are technical and of little economic content. They ensure that under the null hypothesis the normalized sample autocorrelations of $\Delta Y$ are i.i.d. $N(0, 1)$ random variables. [See Hannan and Heyde (1972) for proof.]
requires that moments of orders up to eight all unconditionally exist for output changes. This condition rules out certain types of ARCH models, where it is known that even fourth moments are unbounded.

Under Assumption 1 it is possible to characterize the implications of the random walk null for the entire sequence of deviations of the spectral distribution from a diagonal. The following results are proven in Durlauf (1991a), which builds upon the seminal work of Grenander and Rosenblatt (1953, 1957) on the properties of empirical spectral distribution functions.

Theorem 1. Asymptotic behavior of spectral deviations

Under $H_0$:

(i) $U_T(t) \Rightarrow_w U(t)$, where $U(t)$ is the Brownian bridge on $t \in [0, 1]$.\(^4\)

(ii) $CVM_T = \int_0^1 U_T(t)^2 \, dt \Rightarrow_w \int_0^1 U(t)^2 \, dt = \text{Cramer–von Mises statistic, } CVM$.

(iii) $KS_T = \sup_{t \in [0, 1]} |U_T(t)| \Rightarrow_w \sup_{t \in [0, 1]} |U(t)| \equiv \text{Kolmogorov–Smirnov statistic, } KS$.

(iv) $K_T = \sup_{0 \leq s, t \leq 1} |U_T(t) - U_T(s)| \Rightarrow_w \sup_{0 \leq s, t \leq 1} |U(t) - U(s)| \equiv \text{Kuiper statistic, } K$.

Under any other process which fulfills conditions (ii)–(vii) of the null:

(i) $U_T(t)$ diverges for some fixed $t$.

(ii) $CVM_T, KS_T$, and $K_T$ all diverge.

This theorem provides considerable flexibility in characterizing the properties of output changes. Under the null hypothesis, the normalized deviations of the spectral distribution function from a diagonal, $U_T(t)$, converge to a Brownian bridge. First, one may examine the actual fluctuations of the spectral deviations in order to see whether its sample path is too erratic to be compatible with $H_0$. Two routes are available for looking for deviations from the null. For the entire sample, the $CVM$, $KS$, and $K$ statistics each give a general goodness-of-fit test without specifying a specific set of frequencies to explore. The tests examine the complete deviations of the spectral distribution function from a diagonal. Both long-run and short-run deviations are uncovered at least asymptotically. In this sense, these tests are appropriate when the nature of the alternative is unknown.

Second, for fixed frequencies $\hat{w}$ one may examine whether accumulated variance contributions of $[0, \hat{w}]$ are consistent with white noise. In other words, one can explore the behavior of particular values of $U_T(t)$. This procedure extends the variance ratio tests employed by Cochrane (1988) and Lo and MacKinlay (1988) among others to the frequency domain. To see this recall that

\(^4\) $\Rightarrow_w$ denotes weak convergence. If $B(t)$ denotes a Brownian motion on $[0, 1]$, then the Brownian Bridge is defined as $U(t) = B(t) - tB(1)$. 

the period of \( \sin(\omega t) \) or \( \cos(\omega t) \) equals \( 2\pi/\omega \). Therefore, it is possible to see whether the accumulated variance associated with cycles of different lengths is consistent with white noise, which is analogous to looking for mean reversion over various time horizons.

3. Spectral distribution properties of aggregate output

In this section we explore the properties of aggregate output fluctuations by constructing the spectral distribution function for log per capita output changes. We employ the log of per capita real GNP over the 1870–1989 period. Pre-1947 real GNP figures come from Romer (1989). All other data on real GNP are taken directly from the National Income and Product accounts. The data are analyzed for the entire 1870–1989 period as well as the 1870–1929 and 1947–1989 subperiods in order to control for the effects of the Depression and World War II.

Fig. 1 presents the spectral distribution functions for the three different sample periods. Three features of the diagram stand out. First, there is considerable similarity between the pre-Depression and postwar functions. The entire sample, however, illustrates a substantially different pattern of variance concen-

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Fig. 1. Spectral distribution functions.

5 Throughout all spectral distribution and spectral density function estimates are divided by the sample variance.
tration. In particular, the entire sample period exhibits much greater concentra-
tion of power in the lower frequencies than either of the subsamples. Second, the
spectral distribution functions of both the pre-Depression and postwar periods
appear to be relatively close to the diagonal and thus not strongly at odds with
the null hypothesis. Third, output changes over the entire sample exhibit too
much power in the interval \([0, \pi/2]\) to be consistent with \(H_0\).

3.1. General goodness-of-fit tests

Our initial data analysis considers the consistency of the entire spectral
distribution function with the random walk null. The sample path of the spectral
distribution function defines a continuum of random variables whose fluc-
tuations are asymptotically generated by a Brownian Bridge under \(H_0\). Table 1
provides estimates of the three goodness-of-fit tests, \(CVM_r\), \(KS_r\), and \(K_r\), to
assess whether the sample path is consistent with this distribution.\(^6\) The visual
evidence is confirmed by the hypothesis testing results. All three statistics are
consistent with the null hypothesis for both the pre-Depression and postwar
periods. Over the entire sample, however, all three statistics lead to rejection of
the null. In conjunction with fig. 1, it is apparent that the rejections are
generated by an excess concentration of power in the frequencies \([\pi/4, \pi/2]\),
rather than a deficiency of power in the interval near the zero frequency. The
\(CVM\) results are particularly compelling, as it has been established that the
statistic has excellent size and power properties in finite samples.\(^7\)

These results strongly contrast with the results of other authors, in particular
Cochrane, where the random walk model has been rejected for both the postwar
and the entire sample on the basis of little power at the zero frequency. To
understand why these tests fail to reject the random walk model, it is necessary
to recall that we are treating the complete spectral density of output changes as
a random function and asking whether the sample path realization of the

\(^6\) All distributions of test statistics are computed under the null hypothesis.

\(^7\) In particular, for white noise which is distributed \(N(0, 1)\), the nominal \(CVM\) test size will equal
5% for as few as 40 observations. The inferences in table 1 would be unchanged if one employed
significance levels based upon 4000 independent replications of a 40-observation \(N(0, 1)\) sample. In
this sense, the results of the table are robust to finite sample problems.

As documented in Bernard (1990), the \(CVM\) test exhibits power against many different alternat-
ives. Further, the test performs well in finite samples relative to the Dickey–Fuller and Box–Pierce
tests in detecting some types of deviations from the null. However, sensitivity to all alternatives
creates a cost in terms of finite sample power for certain interesting alternatives. The \(CVM\) test often
has poor power relative to the Dickey–Fuller test if the original series is stationary prior to
differencing. This is the class of alternatives against which the Dickey–Fuller test is consistent.
Bernard therefore concludes that both tests should generally be used. Further analysis of power
properties may be found in Bernard and Durlauf (1990).
random function is consistent with a particular data-generating process – the null hypothesis. Cochrane’s approach examines the consistency of this random function with the null at a single point, the zero frequency. Under our interpretation, the deviations at all frequencies are considered. The spectral shape tests conclude that the deviations at and around the zero frequency do not, when combined with information from other frequencies, constitute statistically significant evidence against the null hypothesis during the pre-Depression and postwar years.

Overall, the appropriate conclusion from table 1 is that rejection of the constant plus white noise model for the pre-Depression and postwar output changes cannot be justified in the absence of a particular prior on the location in the frequency domain of the deviations of the spectral density from a rectangle. The behavior of the sample path of the spectral distribution function for these subsamples is compatible with the null. On the other hand, the null cannot be supported for the entire sample. In the next section, we explore the interpretation of this rejection.

### 3.2. Frequency interval behavior

Our next set of tests explore the behavior of the spectral distribution function at different frequencies. Table 2 provides the values of the spectral distribution function for frequencies sampled at intervals of \( \pi / 8 \). Under the null hypothesis, cycles in the interval \([0, \omega]\) should account for \( \omega / \pi \) of the total variance of output changes. The table compares the accumulated variance for cycles of different lengths with the theoretical values under \( H_0 \). Associated with each \( \omega \) is the normalized deviation \( U_T(t) \) where \( t = \omega / \pi \). As Theorem 1 indicates, under \( H_0 \), \( U_T(t) \sim \mathcal{N}(0, t^2) \) at each fixed value \( t \).

Turning first to the two subsamples, table 2 illustrates that there is some evidence of long-run mean reversion in both the pre-Depression and postwar periods. None of the deviations of the accumulated variance of output changes
Table 2
Spectral distribution function tests. accumulated variance contributions.

<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>$F(\pi/8) = 0.125$</td>
<td>0.06</td>
<td>0.04</td>
<td>0.13</td>
</tr>
<tr>
<td>$(0.37)$</td>
<td>$(0.38)$</td>
<td>$(0.05)$</td>
<td></td>
</tr>
<tr>
<td>$F(\pi/4) = 0.25$</td>
<td>0.14</td>
<td>0.19</td>
<td>0.36</td>
</tr>
<tr>
<td>$(0.59)$</td>
<td>$(0.28)$</td>
<td>$(0.05)$</td>
<td></td>
</tr>
<tr>
<td>$F(3\pi/8) = 0.375$</td>
<td>0.30</td>
<td>0.44</td>
<td>0.64</td>
</tr>
<tr>
<td>$(0.40)$</td>
<td>$(0.30)$</td>
<td>$(2.02)$</td>
<td></td>
</tr>
<tr>
<td>$F(\pi/2) = 0.50$</td>
<td>0.45</td>
<td>0.59</td>
<td>0.70</td>
</tr>
<tr>
<td>$(0.25)$</td>
<td>$(0.39)$</td>
<td>$(1.56)$</td>
<td></td>
</tr>
<tr>
<td>$F(5\pi/8) = 0.625$</td>
<td>0.72</td>
<td>0.69</td>
<td>0.82</td>
</tr>
<tr>
<td>$(0.50)$</td>
<td>$(0.28)$</td>
<td>$(1.47)$</td>
<td></td>
</tr>
<tr>
<td>$F(3\pi/4) = 0.75$</td>
<td>0.85</td>
<td>0.75</td>
<td>0.89</td>
</tr>
<tr>
<td>$(0.55)$</td>
<td>$(0.02)$</td>
<td>$(1.05)$</td>
<td></td>
</tr>
<tr>
<td>$F(7\pi/8) = 0.875$</td>
<td>0.92</td>
<td>0.89</td>
<td>0.95</td>
</tr>
<tr>
<td>$(0.23)$</td>
<td>$(0.06)$</td>
<td>$(0.60)$</td>
<td></td>
</tr>
</tbody>
</table>

* $U_\tau(t)$ statistics in parentheses, $t = \omega/\pi$.

Significant at two-sided asymptotic 5% level.

From the associated theoretical levels, as measured by the $U_\tau(t)$ statistics, is significant at the 5% level. However, there are fairly large deviations of the point estimates from the null. As a result, one could argue that the data in the subsamples speak against the random walk hypothesis, if one is exclusively concerned with long-run mean reversion. Notice, however, neither subsample is characterized by a monotonic shift of power away from low towards high frequencies, as compared to white noise.

Evidence of mean reversion can best be seen in the behavior of $F(\pi/8)$, which captures cycles of 16 years and longer. The point estimates of 0.06 for the pre-Depression and 0.04 for the postwar are both substantially below the 0.125 value predicted by $H_0$. The failure to reject the null reflects the low power of the tests given the small number of observations in each subsample. The point estimates indicate how a mean-reverting process well approximates the data in the low frequencies.

Other frequency ranges in the subsamples also exhibit substantial deviations. Overall, the largest deviation in the pre-Depression period occurs at $\pi/4$ where

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*Strictly speaking, long-run mean reversion refers only to the behavior of the zero frequency. However, differencing a stationary process will induce a deficiency of power in an interval around the zero frequency which will be reflected in the behavior of the spectral distribution function. Also, it is important to observe that a problem for tests near the zero frequency is that the total spectral deficiency is bounded from below, which makes it difficult to uncover long-run mean reversion, especially when the total number of observations is below 30. (These considerations are magnified for point estimates of the spectral density.) We therefore regard the point estimate of $U_\tau(\pi/8)$ as the most helpful way of uncovering the plausibility of mean reversion.*
the actual accumulated variance estimate of 0.14 is below the theoretical level of 0.25. Accumulated variances typically deviate from theoretical levels by values ranging from 8% to 10% for this period. The largest deviation from the null for the postwar sample period occurs at $\pi/2$. An estimated 59% of the total variance of postwar output changes is concentrated in cycles of 5.3 years or more when the theoretical level suggests that 50% of the variance should occur at this frequency. The typical deviation is around 6%.

The estimates over the entire 1870–1989 sample are reported in column 3. Many of the point estimates of the spectral distribution function are statistically significantly different from the levels predicted by the null hypothesis. The point estimates are also substantially different from the white noise levels. For example, the point estimate that 64% of total variance is attributable to frequencies of $3\pi/8$ and lower is far different from the theoretical level of 37.5% and is far larger than the deviations in the two subsamples.

However, all rejections for 1870–1989 are associated with point estimates which exceed the theoretical accumulated variance levels. In fact, all six point estimates are greater than the theoretical levels. Particularly interesting is the juxtaposition of the rejections with the estimate at $\pi/8$. Cycles of sixteen years and longer contribute 13% of total variance which is nearly identical to the theoretical value of 12.5%. As a result, the behavior of output changes over the entire sample cannot be attributed, at least according to these results, to mean reversion. Power in the spectral distribution function is concentrated in the frequencies between $\pi/8$ and $3\pi/8$ relative to the higher frequencies. Consequently, the spectral distribution function lies above the white noise diagonal for all frequencies above $\pi/8$. There is no evidence of a tendency for higher frequencies to dominate output fluctuations relative to low frequencies. The addition of the Depression and war years therefore significantly increases the estimated level of output persistence. [See Perron (1989) for a similar result with a far different interpretation.]

Overall, the data do not speak strongly against the random walk model. It is possible to deduce some evidence against the null from the point estimates. Both the pre-Depression and postwar data exhibit less spectral power in the low frequencies than predicted by the null. This evidence explains why authors such as Cochrane have concluded that there is evidence of mean reversion. The behavior of the entire sample, however, exhibits violations of the null which are not suggestive of long-run mean reversion. While it is certainly possible that the various tests possess low power, there is no compelling evidence against output persistence.

Employing confidence intervals for the different $U_I(t)$ statistics based upon 4000 independent replications of a 40-observation N(0, 1) sample does not affect any of the acceptances or rejections in the table.
4. Spectral density properties of aggregate output

In this section, we explore point estimates of the spectral density of output changes. The literature on mean reversion in output has concentrated on the zero frequency. Fig. 2 and table 3 report Bartlett window estimates of the spectral density based on a lag length of 30. For a lag length of $l$, the Bartlett estimate $\hat{f}_{l,T}(\omega)$ is

$$\hat{f}_{l,T}(\omega) = \sum_{j=-l}^{l} \left(1 - \frac{|j|}{l}\right) \hat{\rho}_Y(j)e^{-ij\omega}. \quad (16)$$

The spectral density estimates allow us to contrast the results of the previous section with the literature on mean reversion which has concentrated on the zero frequency.

From the perspective of statistical inference, the spectral density estimates provide very little information which is contradictory to the null hypothesis. This occurs because the associated standard errors of the Bartlett estimates$^{10}$ are quite large, rendering any interpretation problematic. In fact, for all three data groupings, the standard error of the zero frequency is so large that a point

![Spectral density functions](image)

**Fig. 2.** Spectral density functions.

$^{10}$ Under $H_0$, $\text{var}(\hat{f}_{l,T}(\omega)) = 2l/3T$ if $\omega \in (0, \pi)$, $\text{var}(\hat{f}_{l,T}(\omega)) = 4l/3T$ if $\omega = 0, \pm \pi$. 
estimate of zero is consistent with the white noise null, in the sense that one cannot reject at the 5% level. Any attempt to uncover statistically significant mean reversion through analysis of the zero frequency is therefore not possible. The large standard errors also affect inferences for other frequencies. For example, none of the frequency estimates is inconsistent with white noise in the pre-Depression and postwar eras. The only statistically significant deviations for the null occur over the whole sample for \( f(\pi/8) \) and \( f(\pi/4) \). However, the point estimates are too large rather than too small relative to the null. In total, these results match the earlier spectral distribution function analysis.

The point estimates of the zero frequency are quite small for all three samples. This result parallels Cochrane (1988) and indicates how some long-run mean-reverting processes are compatible with the data, although the mean reversion, if it exists, does not translate into any rejections of the random walk specification.

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Table 3
Spectral density function tests, 30-lag Bartlett window estimates.

<table>
<thead>
<tr>
<th>( H_0: f(\nu) = 1 )</th>
<th>1870-1929</th>
<th>1947</th>
<th>1989</th>
<th>1870-1989</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(0) = 1 )</td>
<td>0.13</td>
<td>0.16</td>
<td>0.29</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.82)</td>
<td>(1.00)</td>
<td>(0.61)</td>
<td></td>
</tr>
<tr>
<td>( f(\pi/8) = 1 )</td>
<td>0.58</td>
<td>0.74</td>
<td>2.12</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.58)</td>
<td>(0.71)</td>
<td>(0.43) b</td>
<td></td>
</tr>
<tr>
<td>( f(\pi/4) = 1 )</td>
<td>0.76</td>
<td>0.79</td>
<td>2.37</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.58)</td>
<td>(0.71)</td>
<td>(0.43) b</td>
<td></td>
</tr>
<tr>
<td>( f(3\pi/8) = 1 )</td>
<td>1.16</td>
<td>2.14</td>
<td>0.81</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.58)</td>
<td>(0.71)</td>
<td>(0.43)</td>
<td></td>
</tr>
<tr>
<td>( f(\pi/2) = 1 )</td>
<td>1.30</td>
<td>1.03</td>
<td>0.66</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.58)</td>
<td>(0.71)</td>
<td>(0.43)</td>
<td></td>
</tr>
<tr>
<td>( f(5\pi/8) = 1 )</td>
<td>1.81</td>
<td>0.50</td>
<td>0.66</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.58)</td>
<td>(0.71)</td>
<td>(0.43)</td>
<td></td>
</tr>
<tr>
<td>( f(3\pi/4) = 1 )</td>
<td>0.63</td>
<td>0.68</td>
<td>0.85</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.58)</td>
<td>(0.71)</td>
<td>(0.43)</td>
<td></td>
</tr>
<tr>
<td>( f(7\pi/8) = 1 )</td>
<td>1.66</td>
<td>0.49</td>
<td>0.30</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.58)</td>
<td>(0.71)</td>
<td>(0.43)</td>
<td></td>
</tr>
<tr>
<td>( f(\pi) = 1 )</td>
<td>0.13</td>
<td>0.30</td>
<td>0.71</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.82)</td>
<td>(1.00)</td>
<td>(0.61)</td>
<td></td>
</tr>
</tbody>
</table>

a Asymptotic standard errors in parentheses.

b Significant at two-sided asymptotic 5% level.
Consequently, table 3 provides no statistically significant reason to reject the random walk null for the pre-Depression and postwar periods, although the point estimates at the zero frequency are quite small. The standard errors associated with the various point estimates are simply too large to permit strong inferences for either subsample or the entire sample. Overall, one cannot conclude that the unit root component of output is small relative to a cyclical component in determining output changes.

5. Summary and conclusions

This paper has explored the second moment properties of annual log per capita output changes for the United States over the last century. The empirical analysis started from the null hypothesis that this output series follows a random walk with drift. A systematic analysis of the spectral distribution function and spectral density of output changes revealed little evidence over the pre-Depression and postwar periods against the null. The sample path of the spectral distribution function of output changes is consistent with white noise. The random walk model appears less plausible than a process exhibiting mean reversion for the subsamples only if one analyses the low frequencies in isolation, although there do not exist statistical rejections of the null. Substantial evidence against the null was found for the entire sample 1870–1989, based upon an excess concentration of power in cycles of 5 to 16 years relative to shorter cycles.

One may certainly attribute most of the results of the statistical analysis to lack of power in the tests. However, the following general result is robust. In the absence of an extremely precise view of the location of the alternative to the random walk null, there is no basis for rejection over the pre-Depression and postwar subsamples. Further, the entire 1870–1989 period exhibits considerable power in the low frequencies. In terms of point estimates, the entire sample exhibits a greater contribution to the variance of total fluctuations by cycles of 5 to 16 years than predicted by a random walk model. Therefore, the data fail to argue against theories of dynamic coordination failure [Durlauf (1991b)] or endogenous technical change [Aghion and Howitt (1989)] which predict substantial output persistence.

References


