PROBABILITY MODELS FOR ECONOMIC DECISIONS by Roger Myerson

excerpts from Chapter 3: Utility Theory with Constant Risk Tolerance

3.1. Taking account of risk aversion: utility analysis with probabilities

In the decision analysis literature, a decision-maker is called risk-neutral if he (or she) is willing to base his decisions purely on the criterion of maximizing the expected value of his monetary income. The criterion of maximizing expected monetary value is so simple to work with that it is often used as a convenient guide to decision-making even by people who are not perfectly risk neutral. But in many situations, people feel that comparing gambles only on the basis of expected monetary values would take insufficient account of their aversion to risks.

For example, imagine that you had a lottery ticket that would pay you either \$20,000 or \$0, each with probability 1/2. If you are risk neutral, then you should be unwilling to sell this ticket for any amount of money less than its expected value, which is \$10,000. But many risk averse people might be very glad to exchange this risky lottery for a certain payment of \$9000.

Given any such lottery or gamble that promises to pay you an amount of money that will be drawn randomly from some probability distribution, a decision-maker's certainty equivalent (abbreviated CE) of this gamble is the lowest amount of money-for-certain that the decision-maker would be willing to accept instead of a gamble. That is, saying that \$7000 is your certainty equivalent of the lottery that would pay you either \$20,000 or \$0, each with probability 1/2, means that you would be just indifferent between having a ticket to this lottery or having \$7000 cash in hand.

In these terms, a risk-neutral person is one whose certainty equivalent of any gamble is just equal to its expected monetary value (abbreviated EMV). A person is risk averse if his or her certainty equivalent of a gamble is less than the gamble's expected monetary value. The difference between the expected monetary value of a gamble and a risk-averse decision-maker's certainty equivalent of the gamble is called the decision-maker's risk premium (abbreviated RP) for the gamble. Thus,

$$RP = EMV - CE.$$

So if a lottery paying \$20,000 or \$0, each with probability 1/2, is worth \$7000 to you, then your risk premium for this lottery is \$10,000 − \$7000 = \$3000.

When you have a choice among various gambles, you should choose the one for which you have the highest certainty equivalent, because it is the one that is worth the most to you. But when a gamble is complicated, you may find it difficult to assess your certainty equivalent for it. The great appeal of the risk-neutrality assumption is that, by identifying your certainty equivalent with the expected monetary value, it makes your certainty equivalent something that is straightforward to compute or estimate by simulation. So what we need now is to find more general formulas that risk-averse decision-makers can use to compute their certainty equivalents for complex gambles and monetary risks.

A realistic way of calculating certainty equivalents must include some way of taking account of a decision-maker's personal willingness to take risks. The full diversity of formulas that a rational decision-maker might use to calculate certainty equivalents is described by a branch of economics called utility theory. Utility theory generalizes the principle of expected value maximization in a simple but very versatile way. Instead of assuming that people want to maximize their expected monetary values, utility theory instead assumes that each individual has a personal utility function that assigns a utility value to every possible monetary income level that the individual might receive, such that the individual always wants to maximize the expected value of his or her utility. For example, suppose that you have to choose among two gambles, where the random variable **X** denotes the amount of money that you would get from the first gamble and the random variable **Y** denotes the amount of money that you would get from the second gamble. A risk-neutral decision-maker would prefer the first gamble if $E(\mathbf{X}) > E(\mathbf{Y})$. But according to utility theory, when U(x) denotes your "utility" for getting any amount of money x, you should prefer the first gamble if $E(\mathrm{U}(\mathbf{X})) > E(\mathrm{U}(\mathbf{Y}))$. (Recall from Chapter 2 that, when **X** has a discrete distribution, the expected value operator $E(\bullet)$ can be written

$$E(\mathrm{U}(\mathbf{X})) = \sum_{\mathrm{x}} \mathrm{P}(\mathbf{X}=\mathrm{x})*\mathrm{U}(\mathrm{x}),$$

where the summation range includes every number x that is a possible value of the random variable **X**.) Furthermore, your certainty equivalent CE of the gamble that will pay the random monetary amount **X** should be the amount of money that gives you the same utility as the expected utility of the gamble. Thus, we have the basic equation

$$\mathrm{U}(\mathrm{CE}) = E(\mathrm{U}(\mathbf{X})).$$

Utility theory can account for risk aversion, but it also is consistent with risk neutrality or even risk-seeking behavior, depending on the shape of the utility function. In 1947, von Neumann and Morgenstern gave an ingenious argument to show that any consistent rational decision maker should choose among risky gambles according to utility theory. Since then, decision analysts have developed techniques to assess individuals' utility functions. Such assessment can be difficult, because people have difficulty thinking about decisions under uncertainty, and because there are so many possible utility functions. But we may simplify the process of assessing a decision-maker's personal utility function if we can assume that the utility function is in some mathematically natural family of utility functions.

For practical decision analysis, the most convenient utility functions to use are those that have a special property called constant risk tolerance. Constant risk tolerance means that, if we change a gamble by adding a fixed additional amount of money to the decision-maker's payoff in all possible outcomes of the gamble, then the certainty equivalent of the gamble should increase by this same amount. This assumption of constant risk tolerance is very convenient in practical decision analysis.

One nice consequence of constant risk tolerance is that it allows us to evaluate independent gambles separately. If you have constant risk tolerance then, when you are going to earn money from two independent gambles, your certainty equivalent for the sum of the two independent gambles is just the sum of your certainty equivalent for each gamble by itself. That is, the lowest price at which you would be willing to sell a gamble is not affected by having other independent gambles. This independence property only holds if you have one of these constant-risk-tolerance utility functions.

If a risk-averse decision-maker's preferences over gambles satisfy this assumption of constant risk tolerance, then the decision-maker must have a utility function U in a simple one-parameter family of functions that are defined by the mathematical formula:

$$U(x) = -EXP(-x/\tau),$$

where the parameter τ is called the risk-tolerance constant. (Here EXP is a standard function in Excel.)

Thus, if we can assume that a decision-maker has constant risk tolerance, then we only need to measure this one risk-tolerance parameter τ for the decision-maker. By asking the decision-maker to subjectively assess his or her personal certainty equivalent for one simple gamble, we can get enough information to compute the risk tolerance that accounts for this personal assessment. Then, once we have found an appropriate risk tolerance for this decision-maker, we will be able to use it to estimate the decision-maker's expected utility and certainty equivalent for any gamble that we can simulate, no matter how complex.

It is natural that this numerical measure of "risk tolerance" may seem mysterious at first. The meaning of these risk-tolerance numbers will become clearer as you get practical experience using them.

	A	B	C	D	E	F	G	H
1	RiskTolerance							
2	36.49	*						
3		EU	U(CE)					
4		-0.9466642	-0.9466642					
5								
6	$X	UTIL(X,RT)	UINV(U,RT)					
7	-20	-1.7299782	-20					
8	-15	-1.5084485	-15					
9	-10	-1.3152864	-10					
10	-5	-1.1468593	-5					
11	0	-1	0					
12	5	-0.8719465	5					
13	10	-0.7602907	10					
14	15	-0.6629328	15					
15	20	-0.578042	20					
16	25	-0.5040217	25					
17	30	-0.4394799	30					
18	35	-0.383203	35					
19	40	-0.3341325	40					
20	45	-0.2913457	45	FORMULAS				
21	50	-0.2540378	50	A2. =RISKTOL(20,-10,2)				
22	55	-0.2215074	55	B4. =-EXP(-(20)/A2)*0.5+-EXP(-(-10)/A2)*0.5				
23	60	-0.1931426	60	C4. =-EXP(-(2)/A2)				
24	65	-0.16841	65	B7. =-EXP(-A7/$A$2)				
25	70	-0.1468445	70	C7. =-$A$2*LN(-B7)				
26				B7:C7 copied to B7:C25.				
27				Chart plots (A7:A25,B7:B25).				
28								
29	*Uses add-in Simtools.xla, available at							
30	http://home.uchicago.edu/~rmyerson/addins.htm							

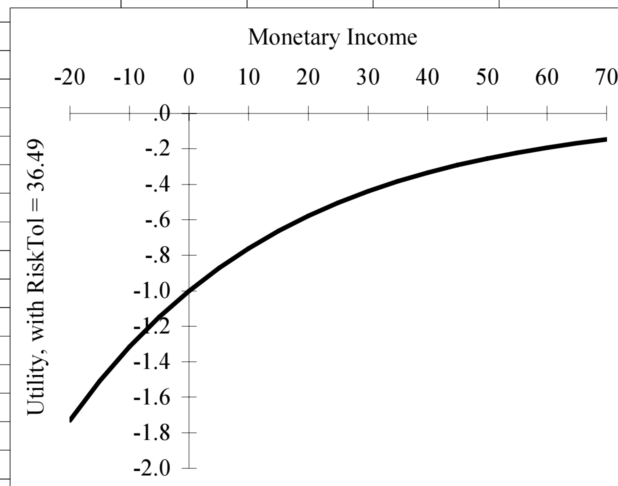

Figure 3.1. A utility function with constant risk tolerance.

...To avoid the confusion of trying to interpret expected utility numbers, we should always convert them back into monetary units by asking what sure amount of money would also yield this same expected utility. This amount of money is then the certainty equivalent of the lottery. Recall basic certainty-equivalent formula U(CE) = *E*U. With constant risk tolerance τ, the utility of the certainty equivalent becomes U(CE) = −EXP(−CE/τ). So the certainty equivalent satisfies −EXP(−CE/τ) = *E*U. But the inverse of the EXP function is the natural logarithm function LN(). So with constant risk tolerance τ, the certainty equivalent of a gamble can be computed from its expected utility by the formula

$$CE = -\tau * LN(-E(U(\mathbf{X}))).$$

3.4. Note on foundations of utility theory

John Von Neumann and Oskar Morgenstern showed in 1947 how simple rationality assumptions could imply that any decision-makers' risk preferences should be consistent with utility theory. In this section, we discuss a simplified version of their argument, which justifies our use of utility theory.

To illustrate the logic of the argument, let us begin with an example. Suppose that we want to study a decision-maker's preferences among gambles that involve possible prizes between (say) \$0 and \$5000. We might start by asking him whether he would prefer to get \$1000 for sure or a lottery ticket that would pay \$5000 with probability 0.20 and would pay \$0 otherwise. If the decision-maker were risk-neutral, he would express indifference between these two alternatives (as they both have expected monetary value \$1000), but suppose that our decision-maker expresses a clear preference for the sure \$1000 over the lottery ticket. Next, we might ask him whether he would prefer the sure \$1000 or the lottery ticket that pays either \$5000 or \$0 if its probability of paying \$5000 were increased to 0.30. Suppose that the decision-maker now responded that he would be willing to give up a sure \$1000 for such a lottery ticket. So somewhere between 0.20 and 0.30 there should be some number p such that this decision-maker would be indifferent between getting \$1000 for sure and getting a lottery ticket that pays \$5000 with probability p and pays \$0 with probability $1-p$. Suppose we ask the decision-maker to think about such lotteries and tell us what probability p would make him just indifferent, and he tells us (perhaps after some long thought) that he would be just indifferent between \$1000 for sure and a lottery ticket that pays \$5000 with probability 0.27 and pays \$0 otherwise.

Next, we might ask this decision-maker a similar question about \$2000 for sure: "For what probability p would you be indifferent between getting \$2000 for sure and a lottery ticket that would pay \$5000 with probability p and \$0 otherwise?" Suppose that, in answer to this question, the decision-maker says that he would be just indifferent between \$2000 for sure and a lottery ticket that pays \$5000 with probability 0.50 and pays \$0 otherwise.

Now consider the following two gambles

Gamble 1 pays \$2000 with probability 0.5 and pays \$1000 otherwise.

Gamble 2 pays \$5000 with probability 0.4 and pays \$0 otherwise.

Given only the above information about this decision-maker, should we be able to predict which of these two gambles he should prefer?

The answer to this question is Yes. If his preferences are logically consistent, he should prefer Gamble 2 here. Let me explain why. Suppose we offered to exchange Gamble 1 for the following two-stage gamble: At stage 1 we toss a fair coin, and if the coin comes up Heads (which has probability 0.5) then we give him a lottery ticket that pays $5000 with probability 0.50 and pays $0 otherwise, but if the coin comes up Tails then we give him a lottery ticket that pays $5000 with probability 0.27 and pays $0 otherwise. He already told us that he would be indifferent between $2000 for sure and the first of these lottery tickets, and he would be indifferent between $1000 for sure and the second of these lottery tickets. So he should be just indifferent between Gamble 1 and this two-stage gamble. But in this two-stage gamble, his ultimate prize is either $5000 or $0, and his probability of getting $5000 is

$$0.5*0.50 + 0.5*0.27 = 0.385$$

(This calculation could be done with a probability tree, as in Section 1.6 of Chapter 1.) Gamble 2 also pays either $5000 or $0, and its probability of paying $5000 is 0.4, which is greater than 0.385. So the decision-maker should rationally prefer Gamble 2 over the two-stage gamble, which he should think is just as good as Gamble 1. Thus he should prefer Gamble 2 over Gamble 1 if he is logically consistent.

This argument can be generalized to show that that logically consistent decision-makers should always satisfy utility theory. To formulate this general argument, we first need to introduce some basic notation. Let us consider a decision-maker who may have to choose among a variety of possible monetary lotteries or gambles. A gamble may be denoted here by a random variable (in boldface) that represents the unknown monetary value that the decision-maker would be paid from this gamble, if he or she accepted it. Given such a decision-maker and given any two gambles **X** and **Y**, we may write

$$\mathbf{X} \succ \mathbf{Y}$$

to denote proposition that "the decision-maker would strictly prefer the gamble **X** over the gamble **Y**." Similarly, we may write

$$\mathbf{X} \sim \mathbf{Y}$$

to denote the proposition that "the decision-maker would be indifferent between the gambles **X** and **Y**."

Utility theory says that each decision-maker's preferences among all possible monetary gambles can be represented by some utility function U(•) such that, for any gambles **X** and **Y**,

$$\mathbf{X} \succ \mathbf{Y} \text{ when } E(\mathrm{U}(\mathbf{X})) > E(\mathrm{U}(\mathbf{Y})),$$

and

$$\mathbf{X} \sim \mathbf{Y} \text{ when } E(\mathrm{U}(\mathbf{X})) = E(\mathrm{U}(\mathbf{Y})).$$

Thus, according to utility theory, once we know a decision-maker's utility function, then we can predict the decision-maker's preferences over all possible monetary gambles. According to utility theory, then, different people may have different preferences for taking and avoiding risks because they have different utility functions.

To keep the argument simple, let us only consider lotteries where the outcome will be selected from some finite set of monetary prizes, which we may denote by

$$\{W_1, W_2, ..., W_n\}.$$

Let W_1 denote the best of these possible outcomes, and let W_n denote the worst of these possible outcomes. Now, for any other possible prize W_i in this set, let us ask the decision-maker:

> "If you were comparing the alternatives of (1) getting W_i dollars for sure, and (2) a binary lottery in which you will get either the best prize W_1 or the worst prize W_n, then what probability of getting the best prize W_1 in this binary lottery would make you just indifferent between these two alternatives."

Obviously, if the probability of the best prize were 1 then the binary lottery would be better; but if the probability of the best prize were 0 then the sure W_i would be better. So there should exist some probability of getting the best prize such that the decision-maker would be just indifferent between these two alternatives.

Given the decision-maker's answer to this question, let $\mathbf{Z_i}$ denote such a random variable that will be equal to either W_1 or W_n with the probability distribution that the decision-maker considers just as good as W_i for sure. That is, let $\mathbf{Z_i}$ be such that

$$P(\mathbf{Z_i}=W_1) + P(\mathbf{Z_i}=W_n) = 1, \text{ and } \mathbf{Z_i} \sim W_i.$$

Notice that we must have

$$P(\mathbf{Z_n}=W_1) = 0,$$

because the worst prize W_n would be worse than any lottery that gave any positive probability of the best prize. Similarly, we must have

$$P(\mathbf{Z_1}=W_1) = 1.$$

Now consider any two general lotteries **X** and **Y** that have outcomes drawn from this finite set of possible prizes $\{W_1,W_2,...,W_n\}$. Since the decision-maker is indifferent between each W_i and the corresponding lottery $\mathbf{Z_i}$, he should not care if we substitute a ticket to the lottery $\mathbf{Z_i}$ wherever he would have gotten the prize W_i. Repeating this substitution argument for each i from 1 to n, we conclude that the lottery **X** should be just as good, for this decision-maker, as a two-stage lottery in which the first stage gives a ticket to a lottery $\mathbf{Z_1}$ or $\mathbf{Z_2}$ or ... or $\mathbf{Z_n}$ respectively with the probabilities $P(\mathbf{X}=W_1)$, $P(\mathbf{X}=W_2)$, ..., $P(\mathbf{X}=W_n)$. In this two-stage lottery, the final outcome will be either the best prize W_1 or the worst prize W_n, and the probability of getting the best prize W_1 is

$$P(\mathbf{X}=W_1)*P(\mathbf{Z_1}=W_1) + P(\mathbf{X}=W_2)*P(\mathbf{Z_2}=W_1) + ... + P(\mathbf{X}=W_n)*P(\mathbf{Z_n}=W_1)$$

So the lottery **X** should be just as good, for this decision-maker, as a lottery that pays either the best prize W_1 or the worst prize W_n, where the probability of the best prize is

$$\sum_{i=1}^{n} P(\mathbf{X}=W_i)*P(\mathbf{Z_i}=W_1).$$

By a similar argument, the decision-maker should be indifferent lottery **Y** should be just as good, for this decision-maker,as a lottery that pays either the best prize W_1 or the worst prize W_n, where the probability of the best prize is

$$\sum_{i=1}^{n} P(\mathbf{Y}=W_i)*P(\mathbf{Z_i}=W_1).$$

Among these two binary lotteries where the only possible outcomes are W_1 or W_n, the better one is obviously the one with the higher probability of the best prize W_1. So the decision-maker should think that **X** is better than **Y** if

$$\sum_{i=1}^{n} P(\mathbf{X}=W_i)*P(\mathbf{Z_i}=W_1) > \sum_{i=1}^{n} P(\mathbf{Y}=W_i)*P(\mathbf{Z_i}=W_1).$$

The trick now is to let $U(W_i) = P(\mathbf{Z_i}=W_1)$. Then we have just shown that

$$\mathbf{X} \succ \mathbf{Y} \text{ whenever } \sum_{i=1}^{n} P(\mathbf{X}=W_i)*U(W_i) > \sum_{i=1}^{n} P(\mathbf{Y}=Wi)*U(W_i).$$

That is, the better gamble should always be the one with the higher expected utility. Thus, we have found a way of assigning utility values to prizes such that the decision-maker, if he is logically consistent, should always prefers the lottery with the higher expected utility.

You may notice that the utility numbers $U(X_i)$ that are generated by this argument are all between 0 and 1. However, there can also be valid utility functions in which the utility numbers are negative, or bigger than 1. In fact, we can multiply a utility function by any positive constant and add any other constant, and we will still have have an equally valid utility function:

Fact 1. Suppose that the decision-maker's preferences over all monetary gambles can be represented by a utility function U(•). Suppose that another function V(•) is an increasing linear transformation of U(•). That is, suppose that there exist two numbers A and B such that A>0 and

$$V(x) = A*U(x) + B, \text{ for all } x.$$

Then V(•) is also a valid utility function for representing this decision-maker's preferences over all monetary gambles.

Proof of Fact 1. For any two gambles **X** and **Y**, if $E(U(\mathbf{X})) > E(U(\mathbf{Y}))$ then

$$E(V(\mathbf{X})) = E(A*U(\mathbf{X})+B) = A*E(U(\mathbf{X})) + B$$
$$> A*E(U(\mathbf{Y})) + B = E(A*U(\mathbf{Y})+B) = E(V(\mathbf{Y})).$$

So comparing expectations of V and U both yield the same preference ordering over gambles.

BASIC FACTS ABOUT RISK AVERSION

Question: Consider an individual whose expresses the risk preferences
[$2000] ~ 0.50[$5000]+0.50[$0] and [$1000] ~ 0.27[$5000]+0.73[$0].
If this individual is rational (consistent), which should he prefer among the lotteries
0.5[$2000]+0.5[$1000] and 0.4[$5000]+0.6[$0]?

Consider an individual with a twice-differentiable utility function u(•).
Suppose this individual has wealth x plus a gamble that will pay a small random amount $\tilde{\varepsilon}$,
such that $E(\tilde{\varepsilon}) = 0$.
Let δ be the maximum that he would pay to insure against this gamble. So $u(x-\delta) = E(u(x+\tilde{\varepsilon}))$.
Assuming that all possible values of $\tilde{\varepsilon}$ are near 0, Taylor series approximations yield
$u(x) - \delta\, u'(x) \approx E[u(x) + u'(x)\,\tilde{\varepsilon} + u''(x)\,\tilde{\varepsilon}^2/2] = u(x) + u''(x)\,\mathrm{Var}(\tilde{\varepsilon})/2$.
and then $\delta \approx -[u''(x)/u'(x)]\,\mathrm{Var}(\tilde{\varepsilon})/2$.
The individual's Arrow-Pratt risk-aversion index at wealth x is $-u''(x)/u'(x)$.
So we find that the individual's value for a small zero-expected-value gamble is approximately half of the variance of the gamble times the individual's risk-aversion index.
Notice that this risk-aversion index r(x) would not change if we changed how i's utility is measured to some other equivalent scale $\hat{u}(x) = Au(x)+B$ where A>0 and B are constants.
The reciprocal of the risk aversion index $\tau(x) = 1/r(x)$ is called the risk tolerance index, which has the advantage of being measured in the same units as money (dollars).

An individual whose risk tolerance is constant, independent of his given wealth x, must have a utility function that satisfies the differential equation $-u''(x)/u'(x) = 1/\tau$, for some constant τ.
This differential equation is equivalent to $d[\mathrm{LN}(u'(x))]/dx = -1/\tau$.
For $\tau>0$, the solutions to this differential equation with $u'>0$ are $u(x) = B - Ae^{-x/\tau}$,
where A>0 and B are arbitary scale constants. We may use B=0 and A=1, to get the canonical constant-risk-tolerance utility function $u(x) = -e^{-x/\tau}$, whose inverse is $x = -\tau\,\mathrm{LN}(-u)$.

Suppose an individual with constant risk tolerance τ, with utility $u(x) = -e^{-x/\tau}$,
will get a random income $\tilde{Y}$ drawn from from some given probability distribution.
The certainty equivalent (CE) of this gamble $\tilde{Y}$ for this individual is the sure amount of money W that individual would be willing to accept instead of this lottery $\tilde{Y}$.
So the certainty equivalent CE satisfies $u(CE) = E(u(\tilde{Y}))$, and so $CE = -\tau\,\mathrm{LN}(-E(u(\tilde{Y})))$

Fact: Suppose u(•) is a utility function with constant risk tolerance, $\tilde{Y}_1$ and $\tilde{Y}_2$ are independent random variables, and $u(W_i) = E(u(\tilde{Y}_i))$ for each i in {1,2}. Then $u(W_1+W_2) = E(u(\tilde{Y}_1+\tilde{Y}_2))$.
Fact: For an individual with constant risk tolerance τ, if $\tilde{Y}$ is drawn from a Normal distribution with mean μ and standard deviation σ then $\tilde{Y}$'s certainty equivalent is $CE = \mu - (0.5/\tau)\sigma^2$.

That is, $-e^{-(\mu-(0.5/\tau)\sigma^2)/\tau} = \int_{-\infty}^{+\infty}\left(-e^{-y/\tau}\right)\frac{e^{-0.5((y-\mu)/\sigma)^2}}{\sqrt{2\pi}\,\sigma}\,dy$. (Proof: use $\int_{-\infty}^{+\infty}\frac{e^{-0.5((y-[\mu-\sigma^2/\tau])/\sigma)^2}}{\sqrt{2\pi}\,\sigma}\,dy = 1$)

EFFICIENT RISK SHARING IN A SYNDICATE

Question: Individuals 1 and 2 have constant risk tolerances T_1 = \$20,000 and T_2 = \$30,000.
1 has an investment paying returns drawn from a Normal distribution: μ=\$35,000, σ=\$25,000.
1 can sell any share θ of this investment to 2 for any price up to her CE of this share.
To maximize 1's total CE, what share θ should 1 offer to sell, for what price? (60% for \$17,250)

Let N denote the set of member of a syndicate or investment partnership.
They hold assets which will yield returns that are a random variable $\tilde{Y}$ with some given probability distribution.
Each individual i in N has a given utility function for money $u_i(\bullet)$.
Let $x_i(y)$ denote the individual i's planned payoff when the syndicate earns $\tilde{Y}=y$.
To be feasible, we must have $\sum_{i\in N} x_i(y) = y,\ \forall y\in\mathbb{R}$.
Let us consider an efficient allocation rule $(x_i(\bullet))_{i\in N}$ that maximizes $\sum_{i\in N} \lambda_i\, E(u_i(x_i(\tilde{Y})))$, subject to this feasibility constraint, for some given positive utility weights $(\lambda_i)_{i\in N}$.

To simplify the characterization of the efficient rule, we assume all appropriate differentiability.
For each outcome y, the marginal weighted utility $\lambda_i u_i'(x_i(y))$ must be equal across all i in N.
That is, there must exist some function $\nu(y)$ such that $\lambda_i\, u_i'(x_i(y)) = \nu(y),\ \forall i\in N,\ \forall y\in\mathbb{R}$ (1).

Differentiating with respect to y, we get $\lambda_i\, u_i''(x_i(y))\, \partial x_i/\partial y = \nu'(y)$ (2).
Equation (1) implies $\lambda_i = \nu(y)/u_i'(x_i(y))$.
Substituting this into (2), we get $[u_i''(x_i(y))/u_i'(x_i(y))]\, \partial x_i/\partial y = \nu'(y)/\nu(y)$.
The bracketted term here is just the Arrow-Pratt index of risk aversion times -1.
Let $\tau_i(x_i)$ denote the reciprocal of the Arrow-Pratt risk-aversion index, which we may call the risk tolerance of individual i when i gets payoff $x_i(y)$. That is $\tau_i(x_i) = -u_i'(x_i)/u_i''(x_i)$.
Then we get $\partial x_i/\partial y = -(\nu'(y)/\nu(y))\, \tau_i(x_i)$.
This derivative $\partial x_i/\partial y$ may be called i's share of the variable risks held by the syndicate.
Feasibility implies $1 = \sum_{j\in N} \partial x_j/\partial y = -(\nu'(y)/\nu(y)) \sum_{j\in N} \tau_j(x_j)$.
Thus we get $\partial x_i/\partial y = \tau_i(x_i(y))/[\sum_{j\in N} \tau_j(x_j(y))]$. (3)
That is, individual i's share of the syndicate's risks should be proportional to i's risk tolerance.

The special case of constant risk tolerance:
Suppose now that each individual i has the utility function $u_i(w) = -e^{-w/T(i)}$,
for some given parameter T(i). Then $u'(w) = (e^{-w/T(i)})/T(i)$, $u''(w) = -(e^{-w/T(i)})/(T(i))^2$.
So the parameter $T(i) = -u'(w)/u''(w)$ is i's constant risk tolerance at all payoff levels.
Then $\partial x_i/\partial y = T(i)/\sum_{j\in N} T(j),\ \forall y$,
and so $x_i(y) = x_i(0) + y\, T(i)/\sum_{j\in N} T(j)$.
That is, individuals who have constant risk tolerance should linearly share all risks in proportion to their risk tolerances.

References: Robert Wilson, "The theory of syndicates," Econometrica 36(1):119-132 (1968).
Karl Borch, "Equilibrium in a reinsurance market," Econometrica 30(3):424-444 (1962).

	A	B	C	D	E	F	G	H	I
1				20	Risk-tolerance parameter (given)				
2	100	x, monetary value		-0.00674	U(x)		100	CE(U)	
3	0.5	sigma			0.000333	U'(x)	0.000341		
4					-1.7E-05	U''(x)			
5					20.00104	Local risk tolerance at x			
6						(estimated numerically)			
7	A .50-.50 lottery of x plus-or-minus sigma.								
8	Prize	Prob'y		Utility					
9	100.5	0.5		-0.00657					
10	99.5	0.5		-0.00691					
11									
12	100	EMV		-0.00674	EU of lottery		99.99375	CE of lottery	
13				0.006249	RP, risk premium				
14									
15				Quadratic approximation:					
16				0.00625	approximate RP		99.99375	approximate CE	
17				5.21E-05	fractional error				
18	FORMULAS								
19	A9. =A2+A3			D12. =SUMPRODUCT(D9:D10,$B$9:$B$10)					
20	A10. =A2-A3			A12. =SUMPRODUCT(A9:A10,$B$9:$B$10)					
21	D2. =-EXP(-A2/$D$1)			D13. =A12-G12					
22	D2 copied to D9:D10			D16. =(0.5/E5)*A3^2					
23	G2. =-$D$1*LN(-D2)			G16. =A2-(0.5/E5)*A3^2					
24	G2 copied to G12			D17. =D16/D13-1					
25	E3. =(D9-D2)/A3								
26	G3. =(D2-D10)/A3								
27	E4. =(E3-G3)/A3								
28	E5. =-AVERAGE(E3,G3)/E4								

Figure: Local risk tolerance

	A	B	C	D	E	F	G	H	I	J	K	L
1	Risky income:			Ms 2	Mr 1				Sum of RiskTolerances			
2	35	mean		30	20	RiskTolerance			50			
3	25	stdev								2's optimal share		
4										0.6		
5	%ile	Income		$ 2 gets	$ 1 gets		2's utility	1's utility		with linear sharing		
6	0.05	-6.12134		-21.418	15.2967		-2.04201	-0.46541		-3.6728		-21.418
7	0.15	9.089165		-12.2919	21.38102		-1.50641	-0.34333		5.453499		-12.2917
8	0.25	18.13776		-6.86292	25.00067		-1.25705	-0.2865		10.88265		-6.8625
9	0.35	25.36699		-2.52132	27.88831		-1.08768	-0.24798		15.22019		-2.52496
10	0.45	31.85847		1.361763	30.4967		-0.95562	-0.21766		19.11508		1.369924
11	0.55	38.14153		5.146229	32.9953		-0.84237	-0.1921		22.88492		5.139765
12	0.65	44.63301		9.04003	35.59298		-0.73983	-0.1687		26.77981		9.034651
13	0.75	51.86224		13.36001	38.50223		-0.64061	-0.14586		31.11735		13.37219
14	0.85	60.91083		18.80887	42.10197		-0.53421	-0.12183		36.5465		18.80135
15	0.95	76.12134		27.92551	48.19583		-0.39422	-0.08983		45.6728		27.92765
16												
17						EU	-1	-0.22792		2 will pay		
18			*	-6.1E-08	29.57526	CE	-6.1E-08	29.57526	*	17.74516	*	3.55E-15
19				3.254829	31.74517	E $			E	21		
20				14.83051	9.887164	Stdev $			Stdev	15		
21				-0.41091	29.30127	estimated CE			CE	17.25		
22					Check U(CE)-EU		0	0				
23	FORMULAS											
24	B6. =NORMINV(A6,$A$2,$A$3)				SOLVER: max H18 by changing D6:D15 subject to G18>=0.							
25	B6 copied to B6:B15				D19. =AVERAGE(D6:D15)				I2. =SUM(D2:E2)			
26	E6. =B6-D6				E19. =AVERAGE(E6:E15)				J4. =D2/I2			
27	E6 copied to E6:E15				D20. =STDEV(D6:D15)				J6. =B6*$J$4			
28	G6. =-EXP(-D6/D$2)				E20. =STDEV(E6:E15)				J6 copied to J6:J15			
29	H6. =-EXP(-E6/E$2)				D18. =CE(D6:D15,D2)				J18. =CE(J6:J15,$D$2)			
30	G6:H6 copied to G6:H15				E18. =CE(E6:E15,E2)				L6. =J6-$J$18			
31	G17. =AVERAGE(G6:G15)				D21. =D19-(0.5/D2)*D20^2				L6 copied to L6:L15			
32	H17. =AVERAGE(H6:H15)				E21. =E19-(0.5/E2)*E20^2				L18. =CE(L6:L15,$D$2)			
33	G18. =-D2*LN(-G17)								J19. =A2*$J$4			
34	H18. =-E2*LN(-H17)								J20. =A3*$J$4			
35	G22. =-EXP(-G18/D2)-G17								J21. =J19-(0.5/D2)*J20^2			
36	H22. =-EXP(-H18/E2)-H17			*The CE function is from Simtools.xla, available at								
37				http://home.uchicago.edu/~rmyerson/addins.htm								

Figure: Optimal risk sharing.