Wait-and-See: Investment Options under Policy Uncertainty

Nancy L. Stokey*

University of Chicago

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Abstract

This paper develops a model of investment decisions in which uncertainty about a one-time change in tax policy induces the firm to temporarily stop investing—to adopt a wait-and-see policy. After the uncertainty is resolved, the firm exploits the tabled projects, generating a temporary investment boom.

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1. INTRODUCTION

This paper develops a model of investment decisions in which uncertainty about a one-time change in tax policy induces the firm to temporarily stop investing—to adopt a wait-and-see policy. The basic idea is that uncertainty about a future tax rate creates uncertainty about the profitability of the investment. If the uncertainty is likely to be resolved in the not-too-distant future, the firm rationally delays committing resources to irreversible projects. After the uncertainty is resolved, the firm exploits the tabled projects, generating a temporary investment boom. The size of the boom depends on the realization of the fiscal uncertainty, with lower realizations of the tax rate producing larger booms.

In the model studied here investment has two inputs, projects and cash. Projects can be thought of as specific investment opportunities, as in McDonald and Siegel (1986) and Jovanovic (2009). For a retail chain or a service provider, projects might be cities or locations where new outlets could be built. For a manufacturing firm, a project might be the construction of a new plant. For a real estate developer, a project might be a parcel of land that could be built on. The key feature of a project is that it is an investment opportunity not freely available to others: it is exclusive to one particular investor, or likely to be so. This feature is important in generating delay: the investor is willing to wait because—at least with high probability—the opportunity will still be available later.

Both projects and liquid assets can be stored, and a "wait-and-see" policy—delay— is defined as a situation where the stocks of both projects and cash are positive. When is delay an optimal strategy? Proposition 4, the main result of the paper, answers this question: an optimal response to uncertainty about future policy *always* involves a period of delay. That period is short if the extent of uncertainty is small, and it does not begin immediately if the uncertainty is in the distant future, but delay is always part of the optimal strategy. Perfectly anticipated policy changes, on the other hand, typically lead investors to accumulate one input or the other, but not both.

Although the literature on investment under uncertainty is vast, most of it focuses on ongoing idiosyncratic shocks, typically interpreted as shocks to the demand for the firm's product or to its cost of production.[1] Most closely related to the model here are papers by Cukierman (1980) and Bernanke (1983). Cukierman looks at the decision problem of an individual firm with a single investment opportunity. The project is characterized by an unknown scale parameter, drawn from a known distribution. Each period the firm receives a signal about the parameter and updates its beliefs. The firm must decide when to invest—how long to wait and receive more information—and how much to invest. The paper shows that an increase in the variance of the distribution from which the parameter is drawn (weakly) increases the number of periods that investment is delayed.

Bernanke (1983) looks at a dynamic inference model, in which investment opportunities arrive every period and the underlying distribution from which these are drawn is, at random dates, replaced with a new one. When this happens, investors learn about it slowly, by observing the outcomes of previous investment decisions. Therefore, after a switch occurs there is likely to be at least one period when investors are very uncertain which distribution is in place. The paper provides an example in which the switch from one distribution to another necessarily produces at least one period in which investors adopt a "wait and see" strategy and no investment takes place.

Other related theoretical work includes papers looking at the aggregate effects of uncertainty through its impact on investment. Hopenhayn and Muniagurria (1996) look at the aggregate implications of policy variability in the sense of regime switching in a standard growth model, and find it can have unexpected growth and welfare

[1] See Dixit and Pindyck (1994) and Stokey (2008) for more detailed discussions of this literature.

effects. More recent work looks at the aggregate effects of increased variance in the distribution of idiosyncratic productivity shocks. In Bloom (2009) the effects of higher variance come through fixed costs of investment, while in Arellano, Bai, and Kehoe (2012) they come through financing constraints. Lee (2012) looks at a setting in which investment opportunities must be created, and higher idiosyncratic volatility has a beneficial effect by inducing investors to create more opportunities and then set a high threshold for selection. By contrast with this work, the model here focuses on uncertainty about a large, one-time policy event.

On the empirical side, related work includes studies of the aggregate effects of fiscal shocks, “news shocks,” and “uncertainty shocks.” Shocks to fiscal policy have long been recognized as an important source of cyclical variation, as in Baxter and King (1993), McGrattan (1994, 2012), and elsewhere. Later work has found evidence that “news shocks” also have aggregate effects, as in Beaudry and Portier (2004, 2006, 2007), Jaimovich and Rebelo (2009), Schmitt-Grohe and Uribe (2010), and Barsky and Sims (2011). Although this work looks at news about productivity growth, it seems plausible that news about fiscal policy would have similar announcement effects.

More recently still, a number of papers have looked for evidence of real effects from “uncertainty shocks,” changes in the variance of the shock of interest. The conclusions here have been mixed, with some studies finding a substantial impact, as in Alexopoulos and Cohen (2009), Fernández-Villaverde, et. al. (2012), Baker, Bloom and Davis (2013), and Bloom, et. al. (2013), and others finding small effects, as in Born, Peter and Pfeiffer (2011), Bachmann and Bayer (2012), and Born and Pfeiffer (2013).

The rest of the paper is organized as follows. Section 2 provides an overview. In section 3 the model is described in more detail, and the transition after the tax change is studied. Section 4 analyzes the firm’s strategy before the tax change. The main result is Proposition 4, which shows that uncertainty about the new tax policy necessarily leads to delay. Section 5 extends the model to allow a Poisson arrival date, and Proposition 5 shows that the main result carries over, provided the arrival rate is not too small. Section 6 contains a numerical example, and section 7 concludes. All proofs are in Appendix A, and Appendix B describes the computational procedure.

2. OVERVIEW

The model uses an investment technology designed to produce an option value. Briefly, the key features are that each investment opportunity is exclusive to the firm, there is an intensity decision that is irreversible, and there are storage possibilities that permit delay. This rest of this section describes these features in more detail.

First, investment requires a project, as well as an input of cash. As noted above, projects should be thought of as specific investment opportunities. For retailers they might be new locations for outlets, for manufacturers they might be new plants, and so on. The key feature of a project is that it is available only to one particular investor. Hence that investor can wait to make a decision about how best to exploit it. This exclusivity assumption could be relaxed to some extent. For example, similar conclusions would hold if projects had a positive hazard rate of becoming available to other investors. The assumption cannot be dropped altogether, however. If a specific project were immediately available to multiple investors, there would be Bertrand-like competition to be the first to exploit it, precluding the possibility of delay.

Second, the intensity of investment in a project is an irreversible decision. Specifically, all investment in a particular project must take place at a single date: the capital cannot be increased or decreased later on. Thus, investment intensity has a putty-clay character: it is flexible ex ante but fixed ex post. This feature could also be relaxed. A model with costly reversibility would deliver similar conclusions, at the cost of added complexity.

Third, the firm can accumulate projects, and those projects do not depreciate.

A positive depreciation rate or a positive probability of becoming obsolete could be incorporated, but storability is key for creating an option value.

Fourth, the total cost of investment is linear in the number of projects for a fixed intensity and strictly convex in the intensity for a fixed number of projects. Strict convexity in the intensity is critical for making projects a valuable commodity. Without it, investment could be concentrated on a small set of projects, at no additional cost.

Finally, the firm cannot borrow, although it can hold stocks of liquid assets. The interest rate on liquid assets is less than the discount rate for dividends, however. Thus, in the absence of uncertainty holding liquid assets is unattractive, and the firm pays out dividends as quickly as possible. But in the presence of uncertainty, liquid assets can be attractive as a temporary investment while waiting for the uncertainty to be resolved. The assumption of a low interest rate on liquid assets makes the firm's dividend policy determinate, and the no-borrowing assumption is primarily for convenience. Allowing the firm to borrow at an interest rate higher than the discount rate would make the firm's financing decision more complex, and reduce or eliminate the incentive to hold liquid assets. Investment decisions would be qualitatively the same however, and the firm would borrow only to accelerate investment.

Formally, time is continuous, and new projects arrive at a constant rate μ. At each date a firm chooses n, the number of projects (a flow), and i, the intensity of investment in each project. Total investment is the product $I = ni$ (a flow). The cost of implementing a project with intensity $i > 0$ is $g(i)$, where the function g is strictly increasing and strictly convex. Hence cost minimization implies that the firm chooses the same intensity for all projects implemented at the same date, and investing at the total rate $I = ni$ has total cost $ng(i) = ng(I/n)$ (a flow) if it is allocated across n projects. If projects and liquid assets have been accumulated, there may also be a discrete investment with intensity $\hat{\imath}$, where the number of projects $\hat{n}$, the increment to capital $\hat{n}\hat{\imath}$, and the total cost $\hat{n}g(\hat{\imath})$ are masses. This type of investment is discussed in more detail later.

The role played by projects can be seen in a two-period example, $t = 1, 2$, with a project inflow of $\mu = 1$ in each period, and k_0 given. The firm chooses the investment scale in each period, $0 \leq n_1 \leq 1$ and $0 \leq n_2 \leq 2 - n_1$, as well as the intensities, $i_1, i_2 \geq 0$. The capital stock and total investment costs are then

$$\begin{aligned} k_t &= (1-\delta)\, k_{t-1} + n_t i_t, \qquad t = 1, 2, \\ TC &= n_1 g(i_1) + \frac{1}{1+\rho} n_2 g(i_2). \end{aligned}$$

If the firm uses the projects as they arrive, choosing $n_1 = n_2 = 1$, total investment costs in the option model are $g(i_i) + g(i_2)/\left(1+\rho\right)$, as usual. But if the firm chooses to concentrate investment in the second period, if $n_1 = 0$ and $n_2 = 2$, with total investment $I_2 > 0$ in the second period, the option model reduces total cost from $g(I_2)$ to $2g(I_2/2)$. Because it allows (forward) smoothing over time, the option model reduces the cost of delay.

Installed capital k produces the net revenue flow $\pi(k)$ and depreciates at the constant rate $\delta > 0$. The revenue from installed capital is taxed at a flat rate τ, and uncertainty about τ is the only risk the firm faces. We will study the effect of uncertainty about a one-time tax reform that is expected in the future. Two assumptions about timing are considered. In the first, the date T of the tax reform is known, and only the new tax rate is uncertain. In the second, the date of the reform is stochastic, with a Poisson arrival time. In both cases the new tax rate $\hat{\tau}$ is drawn from a known distribution $F(\cdot)$.

The following assumptions are maintained throughout:

—dividends are discounted at the constant rate $\rho > 0$;

—liquid assets held by the firm earn interest at the constant rate $0 \leq r < \rho$;

—the firm cannot borrow: all investment is from retained earnings, and the dividend must be nonnegative;

—capital cannot be sold: gross investment must be nonnegative;

—the firm receives a constant flow $\mu > 0$ of new projects;

—the revenue function π is strictly increasing, strictly concave, and twice differentiable, with $\pi(0) = 0$, $\pi'(0) = \infty$, and $\lim_{k\to\infty} \pi'(k) = 0$;

—the cost function g is strictly increasing, strictly convex, and twice differentiable, with $g(0) = 0$, $g'(0) \geq 0$, and $\lim_{i\to\infty} g'(i) = \infty$;

—the time horizon is infinite.

3. THE MODEL AND THE TRANSITION AFTER T

Consider a one-time tax reform, announced at date $t = 0$, that will take effect at date $T > 0$. There are no changes in the tax rate during the interval $[0, T)$, and after the single reform there are no further changes. The new tax rate is not announced at $t = 0$, however. Instead, the firm knows only that it will, at T, be drawn from a known distribution F.

We will compare the firm's optimal strategy in the option model with its behavior in a benchmark model where projects cannot be accumulated. The goal is to characterize the firm's optimal investment on $[0, T)$, in anticipation of the change, and after T, when the new rate is in effect. As usual, it is convenient to start by looking at decisions after T.

In the option model the state variable for the firm is $s = (k, a, m)$, where $k > 0$ and $a, m \geq 0$ are its stocks of capital, liquid assets, and projects. In the benchmark model the state variable is (k, a).

a. The firm's problem at T

At date T the tax rate $\hat{\tau}$ is announced and takes effect. Consider the optimal investment strategy for a firm with state s_T, where $k_T > 0$ and $a_T, m_T \geq 0$. If $a_T, m_T > 0$, the firm's decision problem has two parts. First, it can make a one-time discrete adjustment (DA), using some or all of its stocks of cash and projects to produce an increment to its capital stock. In particular, it can invest in a mass of projects $\hat{n} \geq 0$, with intensity $\hat{\imath} \geq 0$, producing a mass of new capital goods $\hat{I} = \hat{n}\hat{\imath}$. The cost of this investment, $\hat{n}g(\hat{\imath})$, must be financed out of its stock of liquid assets. In addition, the firm can use any remaining liquid assets to pay a discrete dividend $\hat{D}$. After these one-time adjustments, if any, the rest of the firm's decision is a standard (continuous) control problem.

Thus, the firm chooses $\left(\hat{D}, \hat{\imath}, \hat{n}\right)$ and the subsequent dividend flow and investment intensity $\{D(t), i(t), n(t)\}_{t=T}^{\infty}$ to maximize the PDV of total dividends. Let $v(s_T; \hat{\tau})$ denote the maximized value of the firm,

$$v\left(s_T; \hat{\tau}\right) \equiv \max \left[\hat{D} + \int_T^{\infty} e^{-\rho(t-T)} D(t) dt\right] \tag{1}$$

$$\text{s.t.} \quad \begin{aligned} \hat{k}_T &= k_T + \hat{n}\hat{\imath}, \\ \hat{a}_T &= a_T - \hat{D} - \hat{n}g(\hat{\imath}), \\ \hat{m}_T &= m_T - \hat{n}, \\ 0 &\leq \hat{D}, \hat{\imath}, \hat{n}, \hat{a}_T, \hat{m}_T, \end{aligned} \tag{2}$$

$$\begin{aligned} \dot{k} &= ni - \delta k, \\ \dot{a} &= ra + (1 - \hat{\tau})\,\pi(k) - D - ng(i), \\ \dot{m} &= \mu - n, \\ 0 &\leq D, i, n, a, m, \qquad\qquad \text{all } t > T, \end{aligned} \tag{3}$$

where $\hat{s}_T = \left(\hat{k}_T, \hat{a}_T, \hat{m}_T\right)$ denotes the firm's state after the DA.

The benchmark firm faces a similar problem except that it cannot accumulate projects. Hence it cannot make a DA to its capital stock at T, and after T its scale of investment is equal to the inflow of projects at every date. Thus, its problem is as in (1) - (3), but with $m_T = \hat{n} = 0$, and $[n(t) \equiv \mu, \ m(t) \equiv 0, \ \ t > T]$. Let $w\left(k_T, a_T; \hat{\tau}\right)$ denote the maximized value of the benchmark firm.

The option and benchmark models have the same steady state (SS), which is unique and has $a^{ss} = 0$, $m^{ss} = 0$, and

$$\begin{aligned} (1-\tau)\,\pi'\left(k^{ss}\right) &= (\rho+\delta)\,g'(i^{ss}), \\ k^{ss} &= \frac{\mu}{\delta} i^{ss}, \\ D^{ss} &= (1-\tau)\,\pi(k^{ss}) - \mu g(i^{ss}), \end{aligned} \tag{4}$$

where k^{ss}, i^{ss}, and D^{ss} are strictly decreasing in τ. As will be shown below, for any $k_T > 0$ and $a_T, m_T \geq 0$, the solution to (1) - (3) converges asymptotically to the SS.

Before proceeding, however, it is useful to bound the ranges for the capital stock and the tax rate. Let $\overline{k} = k^{ss}(0)$ be the SS capital stock when the tax rate is $\tau = 0$. Only nonnegative tax rates are of interest, so $\overline{k}$ is a natural upper bound on the set of capital stocks. Then define $\overline{\tau} > 0$ as the tax rate for which investment to maintain the capital stock at $\overline{k}$ just exhausts after-tax profits,

$$(1-\overline{\tau})\,\pi(\overline{k}) - \mu g(\delta\overline{k}/\mu) = 0.$$

For any lower tax rate and smaller capital stock, $\tau \in [0, \overline{\tau}]$ and $k \in (0, \overline{k}]$, after-tax revenue is sufficient to finance investment to maintain the capital stock, a fact that simplifies some arguments later on. In the numerical examples in section 6, $\overline{\tau}$ is about 65%.

b. The transition after T in the benchmark model

Propositions 1 - 3 describe the transition after the new tax rate $\hat{\tau}$ is realized. It is convenient first to characterize the solution for the benchmark model, and then describe how the option of storing projects alters that solution.

The benchmark solution has a partial 'bang-bang' form with two critical values for capital. While the capital stock $k(t)$ is below the first critical value, κ^L, all earnings are invested and the dividend is zero. While $k(t)$ is above the second critical value, κ^U, all earnings are paid out as dividends and there is no investment. While $k(t)$ is between the two critical values, both investment and the dividend are positive.

PROPOSITION 1: For any $\hat{\tau} \in [0, \overline{\tau}]$, $k_T \in (0, \overline{k}]$, and $a_T \geq 0$, the solution to the benchmark version of (1)-(3) has the following properties.

(a) The capital stock $k(t)$ converges monotonically to k^{ss}, and $i(t), D(t)$ are continuous along the transition path.

There are two critical values for capital, with $0 < \kappa^L < k^{ss} < \kappa^U \leq \infty$. If $g'(0) > 0$, then $\kappa^U < +\infty$, and if $g'(0) = 0$, then $\kappa^U = +\infty$.

(b) If $a_T = 0$, then $a(t) = 0$, all $t \geq T$, and

for $k(t) \in \left(0, \kappa^L\right)$, $D(t) = 0$ and $i(t) > 0$ is strictly increasing;

for $k(t) \in \left(\kappa^L, k^{ss}\right)$, $D(t) > 0$ is strictly increasing and $i(t) > 0$ is strictly decreasing;

for $k(t) \in \left(k^{ss}, \kappa^U\right)$, $D(t) > 0$ is strictly decreasing and $i(t) > 0$ is strictly increasing;

for $k(t) > \kappa^U$, $D(t) > 0$ is strictly decreasing and $i(t) = 0$.

(c) If $a_T > 0$ and $k_T \geq \kappa^L$, then $\hat{D} = a_T$, $\hat{a}_T = 0$, and the transition is as in (b).

(d) If $a_T > 0$ and $k_T < \kappa^L$, there is a continuous and strictly decreasing function $\alpha(k_T)$, $k_T \in (0, \kappa^L]$, with $\alpha(\kappa^L) = 0$, such that

$\hat{D} = 0$ and $\hat{a}_T = a_T$ for $a_T \leq \alpha(k_T)$;

$\hat{D} = a_T - \alpha(k_T)$, and $\hat{a}_T = \alpha(k_T)$ for $a_T > \alpha(k_T)$.

In either case $a(t)$ is strictly decreasing while $a(t) > 0$, and $a(t)$ reaches zero while $k(t) < \kappa^L$. Thereafter $a(t) = 0$ and the rest of the transition is as in (b).

Let $q = (q_k, q_a)$ denote the costate variables. Figure 1 shows the projection of the phase diagram on (k, q_k) −space, with $a(t) \equiv 0$. The value $\overline{k} = k^{ss}(0)$ is indicated on the horizontal axis. The $\dot{k} = 0$ and $\dot{q}_k = 0$ loci are the dotted curves, and the stable manifold SM is the solid curve. The broken curve, $\chi(k)$, is the threshold where the firm becomes cash constrained. Above that curve the firm is constrained, with $q_a > 1$ and $D = 0$. Below it the firm is unconstrained, with $q_a = 1$ and $D > 0$. The critical value κ^L is defined by the intersection of SM and $\chi(k)$. In this example $\kappa^U > \overline{k}$, so the region where $i^* = 0$ does not appear in the figure.

If the initial asset stock is zero, $a_T = 0$, the transition is along SM in Figure 1, and liquid assets are never acquired. If initial assets are positive, $a_T > 0$, there are two possibilities. If $k_T \geq \kappa^L$, the relevant portion of SM lies below $\chi(k)$, in the region where $q_a = 1$. Thus, the firm's profit flow is sufficient to finance investment at the desired rate with funds left over for a dividend. Hence the entire initial asset stock is paid as a dividend, $\hat{D} = a_T$, and rest of the solution is unchanged.

If $k_T < \kappa^L$, the firm's profit flow is insufficient to finance investment at the desired rate. In this case the initial dividend is less than the asset stock, $0 \leq \hat{D} < a_T$, and the remaining assets are used for investment. Thus, the solution involves $a(t) > 0$ over a finite period. Let $\hat{T}$ denote the date when assets are depleted. Since $\dot{q}_a/q_a = (\rho - r) > 0$, for $t \in [T, \hat{T})$, it follows that $q_a(\hat{T}) > 1$, which implies $k(\hat{T}) < \kappa^L$.

c. The transition after T in the option model

Next consider a firm with the option to store projects. Proposition 2 describes a key feature of the transition: stocks of liquid assets and projects are never held simultaneously. Thus, the DA $(\hat{D}, \hat{\imath}, \hat{n})$ exhausts at least one of the initial stocks, so the subsequent transition begins with $\hat{a}_T = 0$ or $\hat{m}_T = 0$ or both, and at least one stock is zero at every later date as well. In addition, the post-DA stock of liquid assets can be positive only if the post-DA capital stock is less than κ^L.

Proposition 2: For any $\hat{\tau} \in [0, \overline{\tau}]$, $k_T \in (0, \overline{k}]$, and $a_T, m_T \geq 0$, the solution to (1) - (3) has the property that: $\hat{a}_T \hat{m}_T = 0$ and $a(t)m(t) = 0$, all $t > T$. In addition $\hat{a}_T > 0$ implies $\hat{k}_T < \kappa^L$.

The proof of the last claim is immediate. If $\hat{k}_T \geq \kappa^L$ the firm is not liquidity constrained, and $\hat{q}_{aT} = 1$. Hence any excess cash is paid out immediately as a dividend, and $\hat{a}_T = 0$.

The next result describes potential post-DA transitions in the option model. In accord with Proposition 2, attention is limited to initial conditions with $\hat{a}_T \hat{m}_T = 0$, with $\hat{k}_T < \kappa^L$ if $\hat{a}_T > 0$. The transitions involve two new thresholds, κ^0, κ^M. If the capital stock lies outside the interval $\left[\kappa^0, \kappa^M\right]$ and $a = 0$, the firm accumulates projects. For capital stocks below κ^0 the firm is cash constrained, and it accumulates projects in order to fund them later, at higher intensities, after its cash flow has improved. For capital stocks above κ^M the firm is decumulating capital, and it hoards projects to use later to reduce the cost of replacement investment.

Proposition 3: For any $\hat{\tau} \in [0, \overline{\tau}]$, $\hat{k}_T \in (0, \overline{k}]$, $\hat{a}_T, \hat{m}_T \geq 0$, with $\hat{a}_T \hat{m}_T = 0$, and with $\hat{k}_T < \kappa^L$ if $\hat{a}_T > 0$, the solution to (1)-(3), involves two thresholds κ^0, κ^M, in addition to those described in Proposition 1, with $\kappa^0 < \kappa^L < k^{ss} < \kappa^M < \kappa^U$.

(a) If $\hat{a}_T = \hat{m}_T = 0$, then $a(t) = 0$, all $t > T$. For $\hat{k}_T = \left[\kappa^0, \kappa^M\right]$ the solution is as in Proposition 1. For $\hat{k}_T < \kappa^0$ and $\hat{k}_T > \kappa^M$, the transition involves accumulating and then decumulating projects.

(b) If $\hat{m}_T > 0$ and $\hat{a}_T = 0$, then $a(t) = 0$, all $t > T$. The initial stock of projects is exhausted in finite time and remains at zero thereafter. Decumulation begins immediately if $\hat{k}_T \in \left(\kappa^0, \kappa^M\right)$. If $\hat{k}_T < \kappa^0$ or $\hat{k}_T > \kappa^M$, additional projects may be accumulated before decumulation begins.

(c) If $\hat{m}_T = 0$ and $\hat{a}_T > 0$, then $\hat{k}_T < \kappa^L$. The solution has an initial phase during which the entire inflow of new projects is funded, all revenue is used for investment, and the initial stock of assets is gradually drawn down. During this phase, so $n = \mu$, $m = 0$, $D = 0$, and $\dot{a} < 0$. When the stock of liquid assets is exhausted, the rest of the transition is as in (a). In particular, if $k < \kappa^0$ at this point, the solution involves accumulating and then decumulating projects.

If $\hat{m}_T = 0$, the transition in the option model is the same as in the benchmark model for intermediate levels of the capital stock, $k \in \left[\kappa^0, \kappa^M\right]$. Only for capital stocks outside this range does the firm accumulate projects after date T. If $\hat{m}_T > 0$, the firm immediately starts decumulating projects if $\hat{k}_T \in \left[\kappa^0, \kappa^M\right]$. For capital stock outside this range, it accumlates more projects before tapping into the stock.

d. The DA at T

The firm's choice about the DA at T depends on the realized tax rate $\hat{\tau}$, with lower rates producing an incentive for higher investment intensity. For fixed $k_T > 0$ and positive stocks of both assets, $a_T, m_T > 0$, Figure 2 shows, qualitatively, how $\hat{n}, \hat{\imath}, \hat{n}g(\hat{\imath}), \hat{D}$, and the initial costate value $\hat{q}_{aT}$ vary with $\hat{\tau}$. Note that the description of $\hat{\tau}$ as low, moderate or high means relative to other values in the support of F. The initial tax rate τ may be higher or lower than all these values, or anywhere in the middle.

The support of F is divided into three regions. For the lowest realizations of $\hat{\tau}$, Region A, the firm would like to invest in the accumulated projects with a high intensity, but it is cash constrained. In this region the DA exhausts the firm's stock of liquid assets, $\hat{n}g(\hat{\imath}) = a_T$, and some projects remain, $\hat{n} < m_T$. No initial dividend is paid, $\hat{D} = 0$, and cash is at a premium, $\hat{q}_{aT} > 1$. In this region $\hat{n}$ is strictly increasing in $\hat{\tau}$, while $\hat{\imath}$ and $\hat{q}_{aT}$ are strictly decreasing. After date T the stock of remaining projects is gradually used. During this period all earnings are used for investment, and no dividend is paid. When the stock of projects is exhausted the solution lies on $SM(\hat{\tau})$, and the remaining transition is as in the benchmark model.

For higher realizations of $\hat{\tau}$, Region B, the DA continues to use the entire stock of projects, $\hat{n} = m_T$, but the intensity of investment declines and the firm is not cash constrained. The excess liquid assets are paid as an initial dividend, $\hat{D} > 0$. The post-DA state lies on $SM(\hat{\tau})$, and the rest of the transition is as in the benchmark model. Since $\hat{q}_{aT} = 1$, the post-DA capital stock satisfies $\hat{k}_T \geq \kappa^L(\hat{\tau})$.

For even higher realizations of $\hat{\tau}$, Region C, neither the stock of projects nor the stock of liquid assets is exhausted by the discrete investment, and the excess assets are paid as a dividend. That is, $\hat{n} < m_T$, $\hat{D} = a_T - \hat{n}g(\hat{\imath}) > 0$, and $\hat{q}_{aT} = 1$. Indeed, for $\hat{\tau}$ sufficiently large, $\hat{n} = 0$. For tax rates in this region, the post-DA adjustment starts with a stock of projects $\hat{m}_T > 0$, which is gradually used. When the stock is exhausted, (k, q_k) lies on $SM(\hat{\tau})$.

Note that while the intensity $\hat{\imath}$ of the DA is decreasing in $\hat{\tau}$ over the entire range in Figure 2, the scale $\hat{n}$ is increasing in Region A, constant in Region B, and decreasing in Region C. Thus, a stock of projects remains after the DA in Regions A and C. The economic motive for holding investment options after T is different in the two regions, however. In Region A the firm is accumulating capital, but it is cash constrained. Thus, it holds some projects back in order to finance them later, out of retained earnings, at higher intensities. In Region C the firm is reducing its capital stock, and it hoards projects to reduce the cost of replacement investment later on. As we will see below, there must be positive probability of a realization in Region A, where cash is exhausted by the DA and $q_{aT}(\hat{\tau}) > 1$. The firm does not accumulate excessively large stocks of cash.

Note, too, that the firm does not hold liquid assets after date T. Although liquid assets remain after the discrete investment in Regions B and C, they are paid immediately as a dividend.

4. THE FIRM'S STRATEGY ON $[0, T)$

Next consider the firm's strategy during the time interval $[0, T)$. Assume that when the reform is announced at $t = 0$, the firm has no initial stocks of liquid assets or projects, $a_0 = m_0 = 0$, and its initial capital stock is at or below the steady state for the old tax rate, $k_0 \leq k^{ss}(\tau)$. Values for k_0 near $k^{ss}(\tau)$ represent mature firms, while smaller values represent younger firms. The new tax rate $\hat{\tau}$, which takes effect at T, is drawn from the known distribution $F(\hat{\tau})$, and there are no further changes thereafter. If F puts unit mass on a single point, the change is deterministic.

Since there are no initial stocks of liquid assets or projects, there can be no DA or discrete dividend at $t = 0$. Hence the firm chooses $\{(D, i, n)\}_{t=0}^{T}$ to solve

$$\max \int_0^T e^{-\rho t} D(t) dt + e^{-\rho T} \mathrm{E}_{\hat{\tau}} \left[v(s(T); \hat{\tau}) \right] \qquad \text{s.t. } (3), \tag{5}$$

where the expectation uses F. The necessary conditions for an optimum are as before, and the terminal conditions are

$$\begin{aligned} \lim_{t \uparrow T} q_k(t) &= \mathrm{E}_{\hat{\tau}} \left[q_{kT} (\hat{\tau}) \right], \\ \lim_{t \uparrow T} q_x(t) &\geq \mathrm{E}_{\hat{\tau}} \left[q_{xT} (\hat{\tau}) \right], \qquad \text{w/ eq. if } x_T > 0, \qquad x = a, m, \end{aligned} \tag{6}$$

where $q_{xT}(\hat{\tau})$, $x = k, a, m$, are the initial costate values for the problem in (1) - (3), given the initial state $(k_T, a_T, m_T) = (k(T), a(T), m(T))$ and the realized tax rate $\hat{\tau}$. Thus, the costate for capital approaching date T, before the uncertainty is resolved, must equal its expected value ex post. The costates for liquid assets and projects may exceed their expected ex post values if the stock is zero.

a. The period of delay

The main result of the paper is the next proposition, which states that in the option model, uncertainty about the new tax rate always leads to a period of delay: there is an interval of time before T during which investment ceases.

PROPOSITION 4: Suppose a tax change at $T > 0$, drawn from the distribution F, is announced at $t = 0$. Unless F puts unit mass at a single point, there exists $\Delta > 0$ such that $n(t)i(t) = 0$, for $t \in (T - \Delta, T)$.

The proof is by contradiction. Suppose the contrary. Because g is convex, smoothing the intensity of investment across projects reduces the total cost. Delaying some projects from a short period before T until after T permits this type of smoothing. Of course, if the uncertainty is small in magnitude, the period of delay is short. Thus, if T is large, the period of delay may not begin at $t = 0$.

Proposition 4 implies that $a_T, m_T > 0$, so all three conditions in (6) must hold with equality. One important feature of the solution is clear from that fact and Figure 2: the firm's optimal strategy before T necessarily produces a positive probability of being cash constrained when $\hat{\tau}$ is realized. Since liquid assets are acquired before T, the necessary conditions imply that q_a is increasing on $(0, T)$. Since $q_a(0) \geq 1$, it follows that $q_a(T) > 1$. At date T, the post-realization value satisfies $\hat{q}_{aT} > 1$ only if the new tax rate lies in Region A. Hence (6) implies that the solution lies in Region A—where the firm is cash constrained—with strictly positive probability.

Can delay occur in the absence of uncertainty? Yes, anticipation of a deterministic tax decrease can induce the firm to accumulate both cash and projects. The Appendix provides an example of this sort, with $r \approx \rho \approx 0$, an extremely convex cost function g, and an approximately linear profit function π. The motive for delay in this example is simply to wait and exploit projects when the tax climate is more favorable. The assumptions on g and π make the profits from an incremental stock of projects almost independent of when it is exploited, and the assumptions on r and ρ make waiting almost costless.

5. STOCHASTIC ARRIVAL DATE

The date when a tax change will occur may also be uncertain, and in this section the model above is extended to include uncertainty about T. For tractability, the arrival is assumed to be Poisson, with arrival rate θ.

A stochastic arrival date for the tax change does not affect the firm's post-arrival problem in (1)-(3) or the continuation value function $v(s;\hat{\tau})$. Before the arrival the firm's problem, given $s_0 = (k_0, 0, 0)$, is to choose $\{(D, i, n)\}_{t=0}^{\infty}$ to solve

$$\max \int_0^{\infty} e^{-(\rho+\theta)t} \left\{ D(t) + \theta \mathrm{E}_{\hat{\tau}} \left[v(s(t); \hat{\tau}) \right] \right\} dt, \qquad \text{s.t. (3),} \tag{7}$$

where the second term in the objective function is the post-reform continuation value, and the extra exponential term represents the probability that the tax change has not yet occurred.

The solution for (7) consists of $\{[D, i, n, k, a, m], \text{ all } t > 0\}$. This solution converges asymptotically to a steady state. Let $s^*(\theta)$ denote the steady state value for $s = (k, a, m)$, as a function of θ. Thus, if the state is $s^*(\theta)$ and the tax change has not yet occured, the firm does not make any further changes to its stocks.

Let $\widetilde{T}$ denote the random date when the tax change arrives. The initial condition for the post-reform transition depends on the realization of $\widetilde{T}$, call it T_R. For small T_R, the initial condition is close to $s_0 = (k_0, 0, 0)$, so the transition is essentially as in Proposition 1, with negligible initial stocks of assets and projects. For large T_R, the initial condition for the post-arrival transition is close to $s^*(\theta)$.

The next result has two parts. First, it shows that for $\theta > 0$ sufficiently small, the firm does not accumulate stocks of liquid assets or projects, although it may adjust its capital stock slightly. That is, $s^*(\theta) = [k^*(\theta), 0, 0]$, where $k^*(\theta)$ is close to $k^{ss}(\tau)$.

The second part is an analog of Proposition 4. It shows that unless the distribution F puts unit mass on a single point, for all θ is sufficiently large, the steady state stocks of liquid assets and projects are positive, $a^*(\theta) > 0$ and $m^*(\theta) > 0$.

PROPOSITION 5: (a) For all $\theta > 0$ sufficiently small, the steady state for (7) has the property that $s^*(\theta) = [k^*(\theta), 0, 0]$, where $k^*(\theta)$ is close to $k^{ss}(\tau)$. (b) Unless F puts unit mass at a single point, for all θ sufficiently large, $a^*(\theta), m^*(\theta) > 0$.

6. EXAMPLE

The example uses the revenue and cost functions

$$\pi(k) = Ak^{\alpha}, \qquad g(i) = g_1 i + \frac{1}{2} g_2 i^2,$$

and the parameter values

$$\begin{aligned} &A = 1, \quad &&\alpha = 0.70, \quad &&g_1 = 1, \quad &&g_2 = 1.5, \\ &\delta = 0.10, \quad &&\mu = 1, \quad &&\rho = 0.04, \quad &&r = 0.03. \end{aligned}$$

The initial tax rate is $\tau_0 = 0.20$, and the post-reform rate is

$$\hat{\tau} = \begin{cases} \tau^L = 0.22, & \text{with probability } 0.486, \\ \tau^H = 0.42, & \text{with probability } 0.514. \end{cases}$$

Thus, the tax reform could raise the tax rate by 2 or 22 percentage points, with approximately equal probability. The reform is anticipated $T = 0.50$ years in advance. Figures 3 - 5 show the transition for a firm with an initial capital stock that is three quarters of the steady state level for the initial tax rate, $k_0 = 0.75 \cdot k^{ss}(\tau_0)$.

Figure 3 compares the short-run transition in the option model with those in the benchmark model and in a full information setting, where the new tax rate is known at $t = 0$ (although it takes effect at T). With full information, the firm immediately begins adjusting toward the new steady state, in a way that is almost indistinguishable from the adjustment to an immediate tax change. In the benchmark model, the adjustment over $[0, T]$ is, approximately, a weighted average of the two with full information. If the new tax rate is uncertain, the firm adopts a "split-the-difference" investment strategy over $[0, T]$. At date T the path splits, and thereafter the two parts initially parallel the full information paths. The adjustment is lagged, however, reflecting the fact that investment over $[0, T]$ was at a "split-the-difference" level.

In the option model the period of delay begins immediately: investment ceases over $[0, T]$, and the capital stock declines through depreciation. At T, when the uncertainty is resolved, the capital stock jumps because of the discrete adjustment, with the size of the jump depending on the realization of the new tax rate, and thereafter the capital stock adjusts gradually to its new steady state level. The path for the capital stock after T in the option model is very close to the one with full information, reflecting the fact that the investment intensities for the discrete adjustment are tailored to the new tax rate: a higher intensity if the lower tax rate is realized.

Figure 4 displays additional features of the short-run transition. Panels (a)-(h) show the stocks of capital, projects and liquid assets; investment; the marginal values of capital, projects, and liquid assets; and the dividend.

Over $[0, T]$, the capital stock declines through depreciation and the stock of projects grows linearly, as shown in panels (a) and (b). Panel (c) shows liquid asset accumulation, which begins only about 1.5 months before date T, with the stock of assets then growing linearly until T, and panel (d) shows investment, which ceases over $[0, T]$.

The marginal values for capital and projects jump down at $t = 0$, as shown in panels (e) and (f), since news about the tax change—an increase, and perhaps a large one—reduces the value of capital. The marginal value of capital continues to fall over $[0, T]$, as the date of the tax increase draws closer. The marginal value of projects rises at the rate ρ, as it must while the stock is positive.

The marginal value of liquid assets, shown in panel (g), remains constant at unity while dividends are paid, but rises at the rate $\rho - r > 0$ over the period when liquid assets are accumulated. The dividend, in panel (h), jumps up at $t = 0$, when investment ceases, and then jumps down to zero when liquid asset accumulation begins.

At date T the new tax rate is realized, and the discrete adjustment increases the capital stock, as shown in panel (a). If the tax rate is the higher one, τ^H, the firm chooses a lower intensity for investment, and the entire stock of projects can be financed, with some liquid assets left over. Hence the stocks of projects and liquid assets, in panels (b) and (c), both drop to zero. The extra cash is paid as a dividend at T. The marginal values of capital, projects, and liquid assets all drop, with the last falling to unity, as it must when a dividend is paid.

If τ^L is realized, the firm chooses a higher intensity for each project, and it is cash constrained at T. The stock of liquid assets is insufficient to finance the entire stock of projects, and, as shown in panel (b), some projects remain after the discrete adjustment. As shown in panel (d), these are implemented over a short interval after T, as cash becomes available. The marginal values of capital, projects, and liquid assets all increase at T. Until the stock of projects is exhausted, the marginal value of liquid assets remains above unity and no dividend is paid.

Figure 5 displays the longer run transition for the capital stock and investment, as well as the steady state levels (dashed lines) for each realization of the new tax rate. Since k_0 lies between $k^{ss}(\tau^H)$ and $k^{ss}(\tau^L)$, the transition for $\hat{\tau} = \tau^L$ involves accumulating capital, and the transition for $\hat{\tau} = \tau^H$ involves slowly decumulating capital.

In this example the option to store investment projects allows the firm to capture about 42% of the gains from having full information, compared with the benchmark model. Admittedly, those gains are small in this example, but the substantial period of delay is the main point.

7. CONCLUSIONS

The positive predictions of the model developed here are stark: one-time policy uncertainty leads to sharp swings in investment. Although agregate investment also displays sharp swings, additional features will be needed to incorporate this micro-level theory into a quantitative macro model. One possibility is to utilize the fact that some types of policy uncertainty affect only certain sectors or certain types of firms. For example, financial market regulation affects banks, exchange rate policy affects importers, and so on.

Another possibility is to let stored projects depreciate. Depreciation could be interpreted as market changes that make the investment less profitable, or as "poaching" by a rival firm that exploits a similar project. Since projects with high depreciation rates are less storable, for a given policy reform, projects with depreciation rates below some threshold might be stored, while those above the threshold would be exploited immediately. This logic also suggests where to look for delay: since rivals are more important in highly competitive sectors, delay may be more important in sectors with highly differentiated products.

The model here is formulated in terms of tax policy and business fixed investment, but the idea could as well be applied to business hiring decisions and household decisions about purchases of housing and other durables, and to uncertainty about financial regulation, trade policy, energy policy, and other matters that affect the profitability/desirability of various types of investment.

Most of the empirical work to date on uncertainty and investment has focused on the effect of changes in the variance of a shock, and hence that literature sheds little light on the mechanism proposed here. A notable exception is Julio and Yook (2012), who look at cross-country evidence on business investment and election cycles. Consistent with the model here, they find that firms reduce investment substantially during election years, relative to non-election years.

Event studies like those used in finance might also be useful for assessing the importance of policy uncertainty. The event could be a tax or tariff change, a devaluation, or a currency reform. The key predictions of the model are a period of depressed investment before the reform is passed, and a boom after the uncertainty is resolved. And for the reasons noted above, the effect may be stronger in sectors with highly differentiated products.

The mechanism studied here may also be important in prolonging and amplifying the consequences of other shocks. If a financial crisis produces a severe downturn, the private sector may wait for legislative decisions about whether to turn to fiscal stimulus, to see what form it will take and how it will be financed. If the fiscal stimulus is ineffective, investors may wait again for decisions about a second round. If central bankers and political leaders stall on decisions about how to deal with a currency crisis or a potential default on sovereign debt, investors may choose to delay until the main outlines of a policy have been agreed upon.

The contribution of this paper is to highlight the potential for uncertainty about a single policy decision to affect decisions about the timing of investment. Because policy uncertainty affects all firms, it affects aggregate investment, providing a channel through which legislative inaction can lead to a cyclical downturn. Thus, stalling on a major policy issue may have substantial hidden costs. The model here suggests several new avenues for empirical work to assess the role of policy uncertainty in triggering or exacerbating cyclical downturns.

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APPENDIX A: PROOFS OF PROPOSITIONS

Let $q = (q_k, q_a, q_m)$ denote the costates for the problem in (1)-(3). The discrete adjustment $\left(\hat{D}, \hat{\imath}, \hat{n}\right)$ satisfies[2]

$$\begin{aligned}
1 &\leq q_{aT}, && \text{w/ eq. if } \hat{D} > 0, \\
q_{kT} &\leq q_{aT} g'(\hat{\imath}), && \text{w/ eq. if } \hat{n}\hat{\imath} > 0, \\
q_{kT}\hat{\imath} &\leq q_{mT} + q_{aT} g(\hat{\imath}), && \text{w/ eq. if } \hat{n}\hat{\imath} > 0,
\end{aligned} \tag{8}$$

where $q_{kT}(\hat{\tau})$, $q_{aT}(\hat{\tau})$, and $q_{mT}(\hat{\tau})$ are the costate values at date T, after $\hat{\tau}$ is realized. Thereafter the solution satisfies

$$\begin{aligned}
1 &\leq q_a, && \text{w/ eq. if } D > 0, \\
q_k &\leq q_a g'(i), && \text{w/ eq. if } ni > 0, \\
q_k i &\leq q_m + q_a g(i), && \text{w/ eq. if } ni > 0, \qquad \text{all } t > T,
\end{aligned} \tag{9}$$

$$\begin{aligned}
\dot{q}_k &= (\rho + \delta)\, q_k - q_a (1 - \hat{\tau})\, \pi'(k), \\
\dot{q}_a &\leq (\rho - r)\, q_a, && \text{w/ eq. if } a > 0, \\
\dot{q}_m &\leq \rho q_m, && \text{w/ eq. if } m > 0, \qquad \text{all } t > T,
\end{aligned} \tag{10}$$

and the transversality conditions $\lim_{t\to\infty} e^{-\rho t} q_x(t) x(t) = 0$, $x = k, a, m$.

For the benchmark model, $n = \mu$, the solution satisfies the first line in (8) and the first and second lines in (2)-(3) and (9)-(10), and q_m does not appear. Call this the benchmark system.

Proof of Proposition 1: The solution $\hat{D}$, $\{D, i, k, a, q_k, q_a,\ t \geq T\}$ satisfies the benchmark system. Clearly (4) is the steady state, where the assumptions on π and g ensure it is unique.

[2] See Kamien and Schwartz (1991) or Seierstad and Sydsaeter (1977) for a detailed discussion.

Suppose $a_T = 0$, and conjecture the solution has $a(t) \equiv 0$. Then D, i satisfy

$$
\begin{aligned}
q_k &\leq q_a g'(i), && \text{w/ eq. if } i > 0, && (11)\\
D &= (1-\hat{\tau})\,\pi(k) - \mu g(i), && \text{all } t \geq T. && (12)
\end{aligned}
$$

Use (11) and (12) to define $\chi(k)$ by

$$
\mu g\left[g'^{-1}(\chi(k))\right] \equiv (1-\hat{\tau})\,\pi(k).
$$

For any k, the intensity i satisfying $g'(i) = \chi(k)$ is just sufficient to absorb all of after-tax earnings. The function χ is strictly increasing, with $\chi(0) = g'(0)$, and the threshold $\chi(k)$ divides (k, q_k) −space into two regions.

Above the threshold the firm is cash constrained. In this region the dividend is zero and cash is at a premium, $D = 0$ and $q_a > 1$. The investment intensity, call it $i^*(k, q_k)$, is determined by (12), so it is strictly increasing in k and independent of q_k, and (11) determines q_a..

Below the threshold the firm is not cash constrained, so $D > 0$ and $q_a = 1$. In this region the investment intensity is determined by (11), so it is strictly increasing in q_k and independent of k, and the dividend is determined by (12).

Hence the intensity isoquants in (k, q_k) space are L-shaped, with kinks on the $(k, \chi(k))$ threshold. If $g'(0) > 0$, there is a second threshold, the horizontal line where $q_k = g'(0)$. Below this threshold $i^* = 0$, and above it $i^* > 0$.

The locus where $\dot{k} = 0$ satisfies $\mu i^*(k, q_k) = \delta k$, so it is upward sloping, hitting the vertical axis at $q_k = g'(0)$. For $\hat{\tau} \in [0, \overline{\tau}]$ and $k_0 \in (0, \overline{k})$, the $\dot{k} = 0$ locus lies in the region where $D > 0$, below $\chi(k)$. The locus where $\dot{q}_k = 0$ satisfies

$$
(\rho + \delta)\, g'\left[i^*(k, q_k)\right] = (1-\hat{\tau})\,\pi'(k),
$$

so it is downward sloping in the region where $D, i^* > 0$, and vertical in the regions where $D = 0$ and $i^* = 0$.

The stable manifold, call it SM, slopes downward, and for any $k_T \in (0, \overline{k}]$ there is unique value $q_{kT} > 0$ for which the system converges to the steady state. The critical values $\kappa^L < k^{ss} < \kappa^M$ are defined by the points where SM cuts the $\chi(k)$ and $g'(0)$ thresholds. Along SM, intensity i^* increases with k (as q_k falls) for $k < \kappa^L$, it decreases with k for $k \in \left(\kappa^L, \kappa^M\right)$, and it is constant at $i^* = 0$ for $k \geq \kappa^M$. The dividend is $D = 0$ for $k \leq \kappa^L$ and increases with k for $k > \kappa^L$.

To verify that the conjecture $a(t) \equiv 0$ is correct, it suffices to show that $\dot{q}_a / q_a \leq \rho - r$. If $D > 0$, then $q_a = 1$ and $\dot{q}_a = 0$. If $D = 0$, then $k < \kappa^L < k^{ss}$, so k and i^* are increasing, and q_k is decreasing. Since $q_a = q_k / g'(i^*)$, it follows that $\dot{q}_a < 0$.

If $a_T > 0$ and $k_T \geq \kappa^L$, then for $\hat{D} = a_T$ and $\hat{a}_T = 0$, the rest of the solution is as above.

For $a_T > 0$ and $k_T < \kappa^L$, solutions can be constructed as follows. Choose any point (k, q_k) on SM with $k < \kappa^L$, use (12) with $D = 0$ to determine i^*, and calculate $q_a = q_k / g'(i^*) > 1$. Construct trajectories for (k, a, q_k, q_a) by running the relevant ODEs in (3) and (10) backward in time, with $\dot{q}_a / q_a = \rho - r > 0$ and $g'(i^*) = q_k / q_a$. The restriction $q_a \geq 1$ limits the length of the extension. The terminal pairs $(k_T, \alpha(k_T))$ for the longest extensions define the function α.

Varying the length of the backward extension traces out a one-dimensional family of initial conditions (k_T, a_T), and varying the initial point on SM gives a two-dimensional family. Lower initial values for k on SM have higher initial values for q_a, allowing longer extensions. Hence α is a continuous, decreasing function, with $\alpha(k) \rightarrow 0$ as $k \uparrow \kappa^L$.

For $0 < a_T \leq \alpha(k_T)$, the initial dividend is $\hat{D} = 0$, and the transition begins with $\hat{a}_T = a_T$ and $q_{aT} > 1$. The solution follows a constructed trajectory until assets are exhausted. This occurs while $k < \kappa^L$, and thereafter the solution follows SM. For $a_T > \alpha(k_T)$, the initial dividend is $\hat{D} = a_T - \alpha(k_T)$, and the transition begins with $\hat{a}_T = \alpha(k_T)$ and $q_{aT} = 1$. ■

Proof of Proposition 2: For the first claim it suffices to show that $\hat{m}_T > 0$ implies $\hat{a}_T = 0$. Fix $\hat{\tau}$ and suppose $\hat{m}_T > 0$.

If $n(T+z)i(T+z) = 0$ for $z \in (0, \Delta)$, then $\dot{k}(T+z) < 0$, which implies $k > k^{ss}(\hat{\tau})$. In this region SM lies below $\chi(\cdot;\hat{\tau})$, so $q_a(T+z) \equiv 1$, and consequently $a(T+z) \equiv 0$. Hence the solution requires $\hat{a}_T = 0$.

If $n(T+z)i(T+z) > 0$ for $z \in (0, \Delta)$, then the second and third lines in (9) hold with equality over $(T, T+\Delta)$. Differentiate them to get two equations involving $\dot{i}$,

$$\begin{aligned}\frac{\dot{q}_k}{q_k} &= \frac{\dot{q}_a}{q_a} + \frac{g''\dot{i}}{g'},\\ \frac{\dot{q}_m}{q_m} &= \frac{\dot{q}_a}{q_a} + \frac{g''\dot{i}}{g' - g/i}.\end{aligned}$$

By hypothesis $m > 0$, so the third line in (10) holds with equality, and if $a > 0$ the second line also holds with equality, so

$$\begin{aligned}(r+\delta)\,g' - (1-\hat{\tau})\,\pi' &= g''\dot{i},\\ r\,[g' - g/i] &= g''\dot{i},\end{aligned} \tag{13}$$

$$\delta g' + rg/i = (1-\hat{\tau})\,\pi'$$

and hence

$$(1-\hat{\tau})\,\pi' = \delta g' + r\frac{g}{i}.$$

Suppose this condition holds at T. It continues to hold on $(T, T+\Delta)$ if and only if

$$(1-\hat{\tau})\,\pi''\dot{k} = \left[\delta g'' + \frac{r}{i}\left(g' - \frac{g}{i}\right)\right]\dot{i}. \tag{14}$$

The term in brackets on the right is positive, and the second line in (13) implies $\dot{i} \geq 0$. Since $\pi'' < 0$, (14) holds only if $\dot{k} \leq 0$, which implies $k > k^{ss}(\hat{\tau})$. The rest of the argument is as before.

The same argument shows that $m(t) > 0$ implies $a(t) = 0$, for any $t > T$.

Finally, suppose $\hat{k}_T \geq \kappa^L$, and let S^0 denote the solution for $\hat{a}_T = 0$. For $\hat{a}_T > 0$, increasing the dividend by $\Delta\hat{D} = \hat{a}_T$ and using S^0 for the rest of the solution satisfies all of the conditions for an optimum. ■

PROOF OF PROPOSITION 3: (a) Suppose $\hat{a}_T = \hat{m}_T = 0$. The solution for the option model coincides with the solution to the benchmark model if and only if $\dot{q}_m/q_m \leq \rho$, all t.

In the region where $q_a > 1$, we have $ni > 0$ and

$$q_m = q_k \left[i - g/g'\right],$$

so

$$\frac{\dot{q}_m}{q_m} = \frac{\dot{q}_k}{q_k} + \frac{gg''}{g'} \frac{\dot{i}}{ig' - g}.$$

In this region $\dot{q}_k < 0$ and $\dot{k} > 0$. Since the investment intensity i is determined by the cash flow constraint, $\dot{i} > 0$, and the required condition may fail if the growth in intensity is rapid. Define κ^0 as the threshold below which $\dot{q}_m/q_m < \rho$ on SM.

In the region where $ni > 0$ and $q_a = 1$,

$$q_m = ig'(i) - g(i), \qquad \text{and} \qquad g'(i) = q_k.$$

Hence

$$\dot{q}_m = ig''\dot{i} = i\dot{q}_k,$$

so the required condition holds if $k < k^{ss}$. It also holds in some region above k^{ss}, but may fail for k sufficiently large. It may also fail in the region where $ni = 0$. Define κ^M as the threshold above which $\dot{q}_m/q_m > \rho$ on SM.

(b) Solutions with $\hat{m}_T > 0$ can be constructed as follows. Choose $k_S > 0$, let $(k, a, m) = (k_S, 0, 0)$, and let (q_k, q_a, q_m) be the associated costate values on the SM. Construct the solution for (k, m, q_k, q_m) by running the ODEs

$$\dot{k} \;=\; ni - \delta k, \tag{15}$$

$$
\begin{aligned}
\dot{m} &= \mu - n, \\
\dot{q}_k &= q_k \left[\rho + \delta - \frac{(1-\hat{\tau})\,\pi'}{g'} \right], \\
\dot{q}_m &= \rho q_m,
\end{aligned}
\tag{16}
$$

backward in time, with $a(t) = 0$, all t.

If $k_S \leq k^{ss}$, then $D = 0$ and (i, n, q_a) satisfy

$$
\begin{aligned}
\phi(i) &= \frac{q_m}{q_k}, \\
n &= \frac{(1-\hat{\tau})\,\pi(k)}{g(i)}, \\
q_a &= \frac{q_k}{g'(i)},
\end{aligned}
$$

where $\phi(i) \equiv i - g(i)/g'(i)$ is strictly increasing. To verify that $q_a \geq 1$ and $\dot{q}_a/q_a \leq \rho - r$, note that with time running forward, q_m is increasing and q_k is decreasing. Hence $\phi(i)$ and i are increasing, so q_a is falling. Since $q_a \geq 1$ on the SM, both of the required condition holds.

By definition of the thresholds, if $\hat{k}_T < \kappa^0$ or $\hat{k}_T > \kappa^M$, the solution for $\hat{m}_T = 0$ has $n(T+z) < \mu$ for small $z > 0$. Hence the same is true for $\hat{m}_T > 0$ sufficiently small.

If $\hat{k}_T \in \left(\kappa^0, \kappa^M\right)$, the solution for $\hat{m}_T = 0$ has $n(t) = \mu$, all t, and $q_m = q_k \left[i - g/g'\right]$, with $\dot{q}_m/q_m \leq \rho$. Suppose $\hat{m}_T > 0$ is small. Then the solution requires a lower initial value for q_m, a lower intensity, and $n(T+z) > \mu$ for small $z > 0$.

To verify that $\dot{m} = \mu - n < 0$ along the constructed trajectory, recall that $n = \mu$ on the SM. For any fixed k,

$$
\mu g(i^{SM}(k)) \leq (1-\tau)\,\pi(k) = n g(i),
$$

where $i^{SM}(k)$ denotes the intensity on the SM. Hence it suffices to show that $i^{SM}(k) > i$. When the stock of projects is exhausted, the constructed trajectory meets the SM, so $i = i^{SM}(k)$. Before then, i is increasing along the constructed trajectory, and $i^{SM}(k)$ is decreasing along the SM. Hence the required condition holds for all k.

If $k_S \in [k^{ss}, \kappa^M)$, then $q_a = 1$ and $ni > 0$. While $ni > 0$, both the second and third lines in (9) with equality. Together they determine i, from

$$ig'(i) - g(i) = q_m.$$

Since q_m is increasing over time, so is i. Going backward in time, eventually $ni = 0$. While $ni > 0$, the requirement $g'(i) = q_k$ determines n. Specifically, (16) implies

$$\begin{aligned} \frac{\dot{q}_m}{q_m} &= \rho = \frac{g''\dot{i}}{g' - g/i}, \\ \frac{\dot{q}_k}{q_k} &= \rho + \delta - \frac{(1-\hat{\tau})\,\pi'}{g'} = g''\dot{i}. \end{aligned}$$

Combine these conditions to get

$$(1-\hat{\tau})\,\pi'(k) = g'\left[\rho + \delta - \rho\,(g' - g/i)\right].$$

Evidently the term in brackets is positive. Differentiate w.r.t. t to get

$$(1-\hat{\tau})\,\pi''\dot{k} = \left\{g''\left[\rho + \delta - \rho\,(g' - g/i)\right] - \rho g'\left(g'' - g'/i + g/i^2\right)\right\}\dot{i}.$$

Since $k, i, \dot{i}$ are known, and $\dot{k} = ni - \delta k$, this equation determines n. The dividend is the residual from the cash constraint,

$$D = (1-\tau)\,\pi(k) - ng(i).$$

To verify that $\dot{m} = \mu - n < 0$ along the constructed trajectory, recall that $n = \mu$ on the SM. Along the constructed trajectory, q_m, q_k, and i are increasing and k is falling, and the trajectory meets the SM when the stock of projects is exhausted.

There are no solutions of this type with $k_S \geq \kappa^M$.

Varying k_S and the length of the trajectory traces out a two-dimensional manifold of potential initial conditions for $\hat{k}_T, \hat{m}_T$.

(c) If $\hat{a}_T > 0$, then the solution in Proposition 1 is also a solution for the option model if at the date $T + \Delta$ when liquid assets are exhausted, the capital stock satisfies

$k(T+\Delta) \geq \kappa^0$. This happens if $\hat{k}_T$ is not too far below κ^0. Otherwise, the firm is still cash constrained after the initial stock of liquid assets is exhausted, and the solution has a second phase where projects are first accumulated and later used, as in part (a) above. ■

PROOF OF PROPOSITION 4: Suppose to the contrary that for any $\Delta > 0$, the optimal policy has $n(t)i(t) > 0$, all $t \in [T-\Delta, T)$. Consider the following perturbation to the conjectured solution over $(T-\Delta, T+\Delta)$, where $\Delta > 0$ is small. Reduce the flow of projects by $\varepsilon > 0$ over $(T-\Delta, T)$ and accumulate the projects and cash. For $\varepsilon > 0$ sufficiently small, this is feasible. Then increase the flow of projects by ε over $(T, T+\Delta)$, and adjust the intensity on an additional group of projects of size ε. For each $\hat{\tau}$, choose the intensity for this group of 2ε projects as follows. Note that the perturbed path for capital is smooth: there is no jump.

Let $i(T)$ denote the intensity on $(T-\Delta, T)$ and let $i_T(\hat{\tau})$ denote the intensity on $(T, T+\Delta)$, conditional on $\hat{\tau}$. Since Δ is small, these intensities are approximately constant before and after T, although the latter varies with $\hat{\tau}$. For the 2ε projects use the intensity

$$i_P(\hat{\tau}) = \frac{1}{2}\left[i(T) + i_T(\hat{\tau})\right],$$

so the capital stock at $k(T+\Delta)$ is unaltered.

The perturbation changes the investment cost by

$$\Delta_C(\hat{\tau}) = \varepsilon\Delta\left\{2g\left[i_P(\hat{\tau})\right] - g\left[i(T)\right] - g\left[i_T(\hat{\tau})\right]\right\}, \qquad \text{all } \hat{\tau}.$$

Since g is strictly convex, $\Delta_C(\hat{\tau}) \leq 0$, with equality if and only if $i_T(\hat{\tau}) = i(T)$. Unless F puts unit mass at a single point, this condition must fail on a set of $\hat{\tau}$'s with positive probability. Therefore, unless F puts unit mass on a single point,

$$X \equiv \mathrm{E}_{\hat{\tau}}\left\{2g\left[i_P(\hat{\tau})\right] - g\left[i(T)\right] - g\left[i_T(\hat{\tau})\right]\right\} < 0.$$

Since the perturbation reduces the cost of investment, at least weakly, for every $\hat{\tau}$, and delays the timing of expenditures, it is also feasible in the sense that it can be financed without any additional liquid assets.

The cost of the delay is the foregone revenue. The perturbation changes the capital stock by

$$
\begin{aligned}
\Delta_k(T-z) &\approx -\varepsilon\,(\Delta - z)\, i(T), \qquad z \in (0,\Delta)\,, \\
\Delta_k(T+z;\hat{\tau}) &\approx -\varepsilon\Delta i(T) + \varepsilon z\,[2i_P(\hat{\tau}) - i_T(\hat{\tau})] \\
&= -\varepsilon\,(\Delta - z)\, i(T), \qquad z \in (0,\Delta)\,,
\end{aligned}
$$

where the changes after T are conditional on $\hat{\tau}$. Hence the change in revenue is

$$
\begin{aligned}
\Delta_\Pi(\hat{\tau}) &\approx -\left(1-\hat{\tau}\right)\pi' \left[\int_0^\Delta \Delta_k(T-z)dz + \int_0^\Delta \Delta_k(T+z)dz\right] \\
&\approx -2\varepsilon\left(1-\hat{\tau}\right)\pi' i(T) \int_0^\Delta (\Delta - z)dz \\
&= -\varepsilon\Delta^2\left(1-\hat{\tau}\right)\pi' i(T).
\end{aligned}
$$

The reduction in revenue is of order $\varepsilon\Delta^2$, while the reduction in investment costs is of order $\varepsilon\Delta$. As noted above, $X < 0$. Hence for $\Delta > 0$ sufficiently small,

$$
\mathrm{E}_{\hat{\tau}}\left[\Delta_\Pi(\hat{\tau}) - \Delta_C(\hat{\tau})\right] \approx \varepsilon\Delta\left[-\Delta\pi' i(T) - X\right] > 0,
$$

and the perturbation raises expected profits. ■

A deterministic tax cut with delay

To construct an example where a deterministic tax change produces delay, suppose π is approximately linear in the relevant region, let $\rho > 0$ be close to zero, and let $r \approx \rho$. Let $\mu = 1$, and suppose the marginal cost of investment is piecewise linear, with $g'(i) = g_1 i$, for $0 < i \leq 1$ and $g'(i) = g_2 i$, for $i > 1$, with $g_2 >> g_1 = 1$.

We will compare the strategy of investing at the intensity $i = 1$ on $[0, \varepsilon]$, for some $0 < \varepsilon < T$, with the strategy of accumulating projects and cash and carrying out the same investment at T. If the firm invests at the rate $\mu i = 1$ over $[0, \varepsilon]$, then the increment to its capital stock over $[0, \varepsilon]$ is approximately

$$\Delta k(t) \approx \mu i \left(t - \delta t^2/2\right) \approx t, \qquad t \in [0, \varepsilon].$$

Thus, ignoring discounting, the incremental profit over $[0, T]$ is approximately

$$\begin{aligned}
\Delta\Pi_1 &\approx (1-\tau)\,\pi' \left[\int_0^{\varepsilon} \Delta k(t)dt + \Delta k(\varepsilon) \int_{\varepsilon}^{T} e^{-\delta(t-\varepsilon)}dt\right] \\
&\approx (1-\tau)\,\pi' \left[\frac{1}{2}\varepsilon^2 + \varepsilon\frac{1-e^{-\delta(T-\varepsilon)}}{\delta}\right] \\
&\approx (1-\tau)\,\pi'\varepsilon\frac{1-e^{-\delta T}}{\delta}.
\end{aligned}$$

The increment to the capital stock at date T from this investment is $\Delta k_1(T) \approx \varepsilon e^{-\delta T}$.

If the firm waits until date T, it gets no incremental profit flow over $[0, T]$, but the incremental capital stock at T is $\Delta k_2(T) = \varepsilon$. Hence the extra capital if investment is delayed is $\Delta x \equiv \Delta k_2 - \Delta k_1 = \varepsilon\left(1 - e^{-\delta T}\right)$. The extra profit flow from this increment from T onward is approximately

$$\begin{aligned}
\Delta\Pi_2 &\approx (1-\hat{\tau})\,\pi' \int_T^{\infty} \Delta x e^{-\delta(t-T)}dt \\
&= (1-\hat{\tau})\,\pi'\varepsilon\left(1-e^{-\delta T}\right)\frac{1}{\delta}.
\end{aligned}$$

The firm chooses to delay investment if $\Delta\Pi_2 > \Delta\Pi_1$, which holds for any tax cut, $1 - \hat{\tau} > 1 - \tau$.

Stochastic arrival date

The first order conditions for the problem in (7) are again as in (9), but the laws of motion for the costates now include terms that pick up the expected capital gains or losses on the assets when the tax change occurs. Thus, (10) is replaced by

$$\dot{q}_k = (\rho+\delta)\,q_k - q_a\,(1-\tau)\,\pi'(k) + \theta\left\{q_k - \mathrm{E}_{\hat{\tau}}\left[q_{kT}(s;\hat{\tau})\right]\right\}, \tag{17}$$

$$
\begin{aligned}
\dot{q}_a &\leq (\rho - r)\, q_a + \theta \left\{ q_a - \mathrm{E}_{\hat{\tau}} \left[q_{aT}(s;\hat{\tau}) \right] \right\}, \qquad \text{w/ eq. if } a > 0, \\
\dot{q}_m &\leq \rho q_m + \theta \left\{ q_m - \mathrm{E}_{\hat{\tau}} \left[q_{mT}(s;\hat{\tau}) \right] \right\}, \qquad \text{w/ eq. if } m > 0,
\end{aligned}
$$

where $q_{xT}(s;\hat{\tau})$ denotes the initial value of the costate for the post-reform transition, conditional on the state s and the realized tax rate $\hat{\tau}$, and where we have used the fact that $v_x(s;\hat{\tau}) \equiv q_{xT}(s;\hat{\tau})$, $x = k, a, m$.

From (3), a steady state requires

$$
n^* = \mu, \qquad i^* = \delta k^*/\mu, \qquad D^* = (1-\tau)\,\pi(k^*) - \mu g(i^*) + r a^*, \tag{18}
$$

where the restriction to tax rates $\tau \in [0, \overline{\tau})$ implies $k^* \leq k^{ss}(0)$, so $D^* > 0$. Since $D^*, n^*, i^* > 0$, (9) implies

$$
q_a^* = 1, \qquad q_k^* = g'(i^*), \qquad q_m^* = \phi(i^*), \tag{19}
$$

where $\phi(i) \equiv i g'(i) - g(i)$, with $\phi'(i) > 0$. These conditions determine i^*, q_k^*, q_m^* as functions of k^*. Use (18) and (19) to find that at a steady state (17) requires

$$
\begin{aligned}
\theta \left\{ \mathrm{E}_{\hat{\tau}} q_{kT}(s^*;\hat{\tau}) - g'(i^*) \right\} &= (\rho + \delta)\, g'(i^*) - (1-\tau)\,\pi'(k^*), \\
\theta \left\{ \mathrm{E}_{\hat{\tau}} \left[q_{aT}(s^*;\hat{\tau}) \right] - 1 \right\} &\leq \rho - r, \qquad \text{w/ eq. if } a^* > 0, \\
\theta \left\{ \mathrm{E}_{\hat{\tau}} \left[q_{mT}(s^*;\hat{\tau}) \right] - \phi(i^*) \right\} &\leq \rho \phi(i^*), \qquad \text{w/ eq. if } m^* > 0.
\end{aligned} \tag{20}
$$

The three conditions in (20) determine (k^*, a^*, m^*). Let $s^*(\theta)$ denote the SS as a function of θ.

Proof of Proposition 5: (a) For $\theta = 0$, the first line in (20) requires $k^* = k^{ss}(\tau)$. Then for $a^* = m^* = 0$, the second and third lines hold with strict inequality. Hence $s^*(0) = \left[k^{ss}(\tau), 0, 0 \right]$.

Consider the term in braces on the left in the first line of (20),

$$
X \equiv \mathrm{E}_{\hat{\tau}} \left[q_{kT}(s^*(0);\hat{\tau}) \right] - g'(i^*).
$$

If $X = 0$, then the first line also holds for $\theta > 0$.

Otherwise, since g is strictly convex and π is strictly concave, an increase in k^* increases the RHS of the first line and decreases the LHS. Thus, if $X > 0$, an increase in k^* is needed to restore equality for small $\theta > 0$. If $X < 0$, then by the same reasoning, a decrease in k^* is needed to restore equality. In both cases the second and third lines continue to hold for $\theta > 0$ sufficiently small. Hence, in these cases $s^*(\theta) = [k^{ss}(\tau) + \varepsilon(\theta), 0, 0]$, where $\varepsilon(\theta)$ has the sign of X.

(b) Choose θ large, and suppose to the contrary that $a^* = 0$ or $m^* = 0$ or both. Consider the initial condition $s_0 = s^*(\theta)$, and consider the following perturbation to the strategy of choosing $s(t) = s^*(\theta)$, all $t > 0$.

Let i^* denote the SS intensity, and choose $\varepsilon, \Delta > 0$ small. Over $(0, \Delta)$, reduce the flow of projects by ε, keeping the intensity unchanged. At $t = \Delta$, the capital stock is reduced by $\varepsilon \Delta i^*$, and the firm has a stock of $m = \varepsilon \Delta$ untapped projects and a stock of $a = \varepsilon \Delta g(i^*)$ liquid assets. Over $\left(\Delta, \widetilde{T}\right)$ reduce the intensity of replacement investment by $\varepsilon \Delta i^* \delta / \mu$, so the capital stock remains constant. Pay the interest on the accumulated liquid assets and the savings in replacement cost as dividends. The EDV of the additional dividends is

$$\begin{aligned} \Delta_D &= \varepsilon \Delta \left[r g(i^*) + \frac{\delta}{\mu} i^* g'(i^*) \right] \int_0^\infty e^{-(\rho+\theta)t} dt \\ &= \left[r g(i^*) + \frac{\delta}{\mu} i^* g'(i^*) \right] \frac{\varepsilon \Delta}{\rho + \theta}. \end{aligned} \tag{21}$$

These terms are positive and have order $\varepsilon \Delta$.

After the tax change arrives, over $\left(\widetilde{T}, \widetilde{T} + \Delta\right)$ increase the scale of investment by ε and alter the intensity for an additional ε projects as in the proof of Proposition 4, so the capital stock at $\widetilde{T} + \Delta$ is as it would have been under the original plan. Define $i_T(\hat{\tau})$ and $i_P(\hat{\tau})$ as in the proof of Prop. 4. Conditional on the new tax rate $\hat{\tau}$, the perturbation to the investment cost is

$$\Delta_C(\hat{\tau}) = \varepsilon \Delta \left\{ 2g\left[i_P(\hat{\tau})\right] - g(i^*) - g\left[i_T(\hat{\tau})\right] \right\}, \qquad \text{all } \hat{\tau}.$$

As shown in the proof of Prop. 4, $\Delta_C(\hat{\tau}) \leq 0$, with equality if and only if $i_T(\hat{\tau}) = i^*$. Therefore, unless F puts unit mass on a single point,

$$X(i^*) \equiv \mathrm{E}_{\hat{\tau}} \left\{ 2g\left[i_P(\hat{\tau})\right] - g(i^*) - g\left[i_T(\hat{\tau})\right] \right\} < 0.$$

Hence this contribution of the perturbation to the EDV of profits is

$$\begin{aligned} -\Delta_C &= -\varepsilon \Delta X(i^*) \int_0^\infty \theta e^{-(\rho+\theta)t} dt \\ &= -\theta X(i^*) \frac{\varepsilon \Delta}{\rho + \theta}. \end{aligned} \tag{22}$$

This term is positive and has order $\varepsilon\Delta$.

The perturbation changes the capital stock by

$$\Delta_k(t) = \begin{cases} -\varepsilon i^* t, & t \in (0, \Delta), \\ -\varepsilon i^* \Delta, & t \in \left(\Delta, \widetilde{T}\right), \\ \varepsilon i^* \left[t - \left(\widetilde{T} + \Delta\right)\right], & t \in \left(\widetilde{T}, \widetilde{T} + \Delta\right). \end{cases}$$

The PDV of the change in revenues over the interval $(0, \Delta)$, evaluated at $t = 0$, is

$$\begin{aligned} \Delta_{Ra} &= -(1-\tau)\, \pi' \varepsilon i^* \int_0^\Delta t e^{-\rho t} dt \\ &\approx -(1-\tau)\, \pi' i^* \varepsilon \frac{\Delta^2}{2}, \end{aligned}$$

which has order $\varepsilon\Delta^2$. The EDV of the change in revenues over $\left(\widetilde{T}, \widetilde{T} + \Delta\right)$ also has order $\varepsilon\Delta^2$. Hence both terms can be dropped.

The EDV of the change in revenue over $\left(\Delta, \widetilde{T}\right)$ is

$$\begin{aligned} \Delta_{Rb} &= -\varepsilon \Delta i^* (1-\tau)\, \pi' \int_\Delta^\infty e^{-(\rho+\theta)} \theta dt \\ &\approx -(1-\tau)\, \pi' i^* \frac{\varepsilon \Delta}{\rho + \theta}. \end{aligned} \tag{23}$$

This term is negative and has order $\varepsilon\Delta$.

Summing the components in (21)-(23) and dropping those of order higher than $\varepsilon\Delta$, we find that the perturbation is profitable for Δ sufficiently small if and only if

$$0 < rg(i^*) + \frac{\delta}{\mu} i^* g'(i^*) - \theta X(i^*) - (1-\tau)\,\pi' i^*.$$

The first three terms are positive, and the last is negative. But as θ grows without bound, with F fixed, k^* and $i^* = k^*\delta/\mu$ converge to limiting values. Hence for θ sufficiently large, the third term, which is positive, dominates the last one. ■

APPENDIX B: COMPUTATIONAL METHOD

This Appendix describes the computational method for the example.

1. Preliminary: Choose the three tax rates, and construct the stable manifolds $SM(\tau)$, for $\tau = \tau_0, \tau^\ell, \tau^h$, including values for the costate q_m. Choose an initial capital stock $k_0 \leq k^{ss}(\tau_0)$ and probabilities θ^ℓ, θ^h, for the two post-reform tax rates.

2. Before T, acquiring projects and liquid assets: Choose T not too large, and conjecture that $n(t) = 0$ on $(0, T)$. Then

$$k_T = k_0 e^{-\delta T}, \qquad m_T = \mu T.$$

Since $r < \rho$, the firm does not pay a dividend while it holds liquid assets. Thus, for some date $S \in (0, T)$, all earnings are paid as dividends before S, and all are retained after S. Given S, the stock of liquid assets and its costate at T are

$$\begin{aligned} a_T &= (1-\tau_0) \int_S^T \pi\left(k_0 e^{-\delta t}\right) e^{r(T-t)} dt > 0, \\ q_{aT} &= e^{(\rho - r)(T-S)} > 1. \end{aligned}$$

Thus, increasing S reduces both a_T and q_{aT}. An iterative procedure determines S.

Conjecture that the tax rates $\hat{\tau} = \tau^\ell$ and $\hat{\tau} = \tau^h$ lie in Regions A and B of Figure 2, respectively. In Region B the firm has excess liquid assets, which are paid out as a discrete dividend at T. Hence in this region $\hat{q}_{aT}(\tau^h) = 1$. In Region A the firm is cash-constrained and $\hat{q}_{aT}(\tau^\ell) > 1$.

3. The solution for $\hat{\tau} = \tau^\ell$: In Region A the stock of liquid assets is exhausted by the DA, but some projects remain. The solution is constructed by backward shooting. Choose a candidate terminal point $\left(k^\Delta, q_k^\Delta\right)$ on $SM(\tau^\ell)$ and a candidate value $\Delta_T > 0$ for the length of time needed after the DA to use up the residual stock of projects and reach the point $\left(k^\Delta, q_k^\Delta\right)$. The costate for projects at that point, call it q_m^Δ, is known, and no projects are held. Construct trajectories for (i, n) and (k, m, q_k, q_m) by running backward from $\left(k^\Delta, 0, q_k^\Delta, q_m^\Delta\right)$, for Δ_T units of time, the system of ODEs

$$\begin{pmatrix} \dot{k} \\ \dot{m} \\ \dot{q}_k/q_k \\ \dot{q}_m/q_m \end{pmatrix} = -\begin{pmatrix} ni - \delta k \\ \mu - n \\ \rho + \delta - \left(1-\tau^\ell\right)\pi'(k)/g'(i) \\ \rho \end{pmatrix}, \tag{24}$$

with investment satisfying

$$\begin{aligned} i - \frac{g(i)}{g'(i)} &= \frac{q_m}{q_k}, \\ n &= \frac{\left(1-\tau^\ell\right)\pi(k)}{g(i)}. \end{aligned} \tag{25}$$

The endpoint from this exercise is a candidate for the post-DA state $\left(\hat{k}_T, \hat{m}_T\right)$. Let $\hat{\imath}$ be the investment intensity at this point, and let $\hat{n} = m_T - \hat{m}_T$. Check whether the DA is consistent with these values, whether

$$\hat{k}_T - k_T = \hat{n}\hat{\imath}, \qquad \text{and} \qquad a_T = \hat{n}g(\hat{\imath}).$$

Adjust the candidate point $\left(k^\Delta, q_k^\Delta\right)$ on $SM(\tau^\ell)$ and the time interval Δ_T until both conditions are satisfied.

To verify that q_a satisfies the first line of (9) and the second line of (10), note that with time running forward, q_m is rising and q_k is falling, so i is rising. Hence

$q_a = q_k/g'(i)$ is falling, so $\dot{q}_a/q_a < \rho - r$. In addition, since $q_a \geq 1$ on $SM(\tau^\ell)$, it follows that $q_a \geq 1$ on all of $(T, T+\Delta_T)$.

The rest of the transition to the new steady state follows the stable manifold $SM(\tau^\ell)$.

4. The solution for $\hat{\tau} = \tau^h$: In Region B the stock of projects is exhausted by the DA. Let $i^{SM}(k;\tau^h)$ denote the investment intensity on $SM(\tau^h)$, given the capital stock k. Since the intensity does not jump after the DA, choose the intensity $\hat{\imath}$ satisfying

$$\hat{\imath} = i^{SM}(k_T + m_T \hat{\imath}; \tau^h).$$

Since i^{SM} is decreasing in its first argument, there is a unique solution. The stock of liquid assets at T must finance this investment, which requires $a_T \geq m_T g(\hat{\imath})$. The remaining liquid assets are paid out as a discrete dividend at T.

After the DA, the rest of the transition to the new steady state follows the stable manifold $SM(\tau^h)$.

5. Iterating to find S: Under the conjecture that τ^ℓ and τ^h lie in Regions A and B of Figure 2, $\hat{q}_{aT}(\tau^\ell) > 1$ and $\hat{q}_{aT}(\tau^h) = 1$. A larger initial stock of liquid assets a_T reduces $\hat{q}_{aT}(\tau^\ell)$, so $\mathrm{E}_{\hat{\tau}}\left[\hat{q}_{aT}\right]$ is a decreasing function of a_T. Hence $\mathrm{E}_{\hat{\tau}}\left[\hat{q}_{aT}\right]$ increasing in S, while q_{aT} is decreasing in S, and there is at most one value for which $\mathrm{E}_{\hat{\tau}}\left[\hat{q}_{aT}\right] = q_{aT}$. If no solution exists, the conjecture that $n(t) = 0$ on $(0,T)$ is incorrect, and a smaller value for T is needed, or else the conjecture that τ^ℓ lies in Region A is incorrect.

6. The costates before T: The post-DA costates $\hat{q}_{kT}(\hat{\tau})$ and $\hat{q}_{mT}(\hat{\tau})$ are known, for $\hat{\tau} = \tau^\ell, \tau^h$. Use (6) to determine q_{kT} and q_{mT}, and then use the laws of motion in (10), running backward, to construct solutions over $(0,T)$.

Figure 1: phase diagram for the benchmark model, a = 0

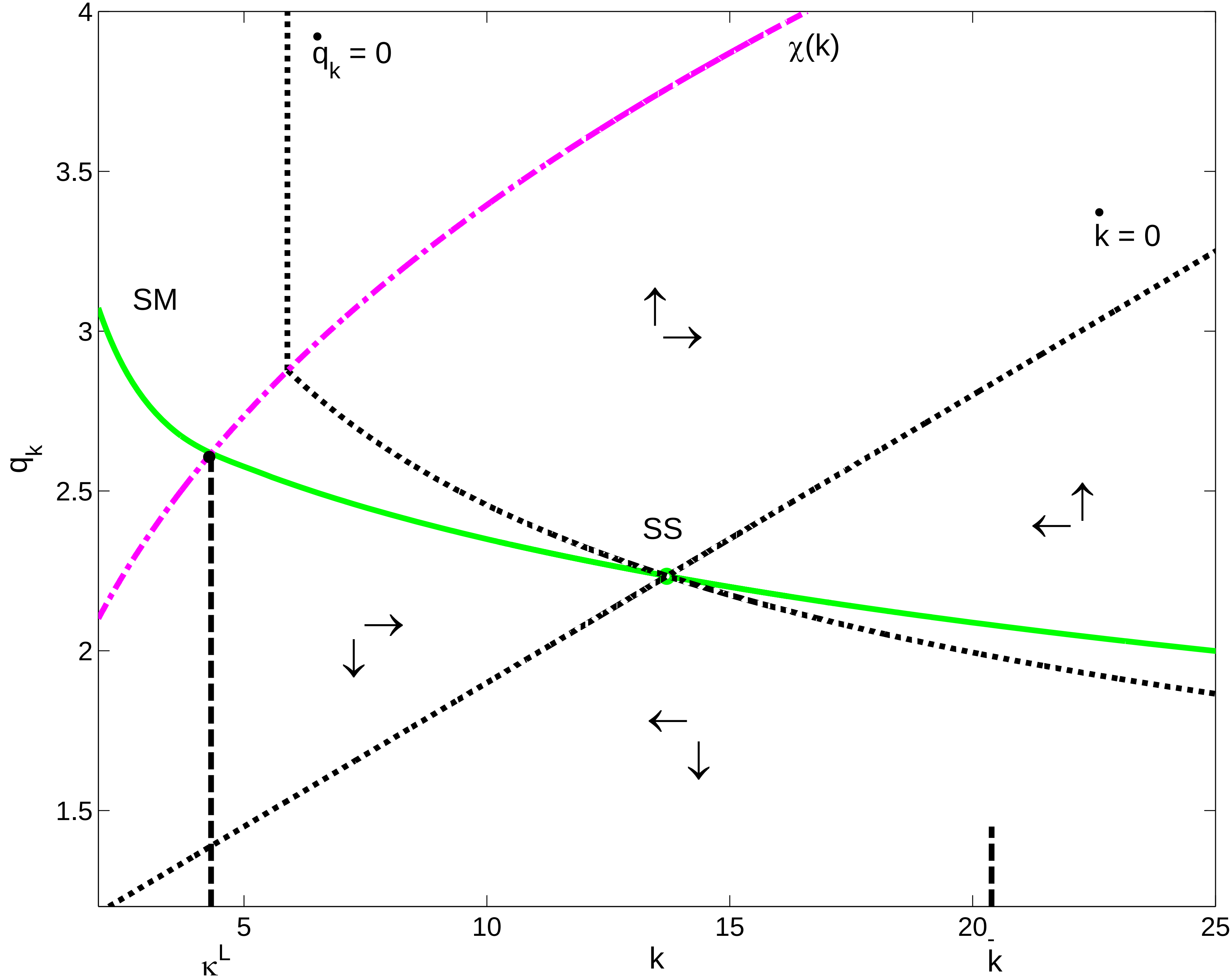


Figure 2: the Discrete Adjustment

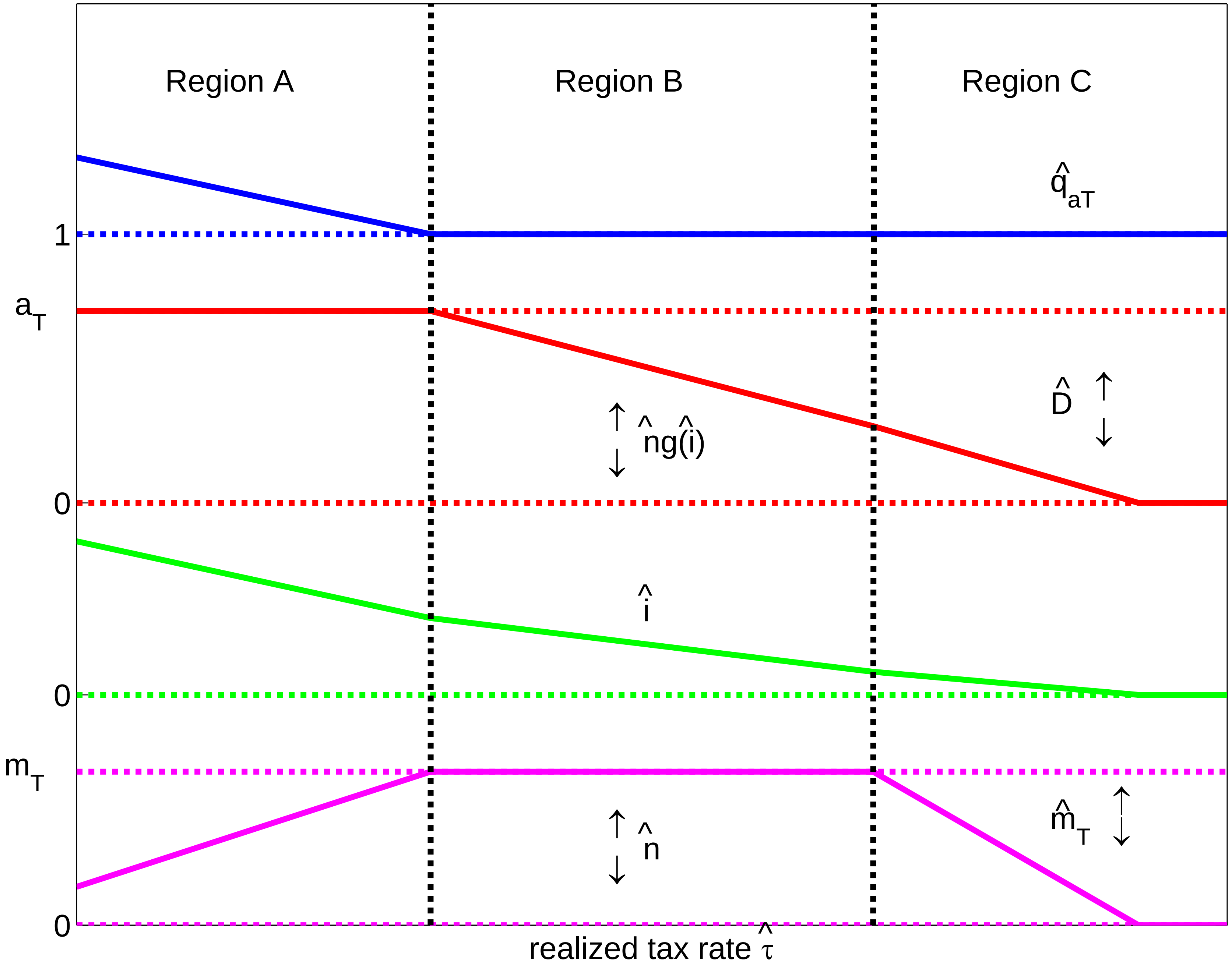
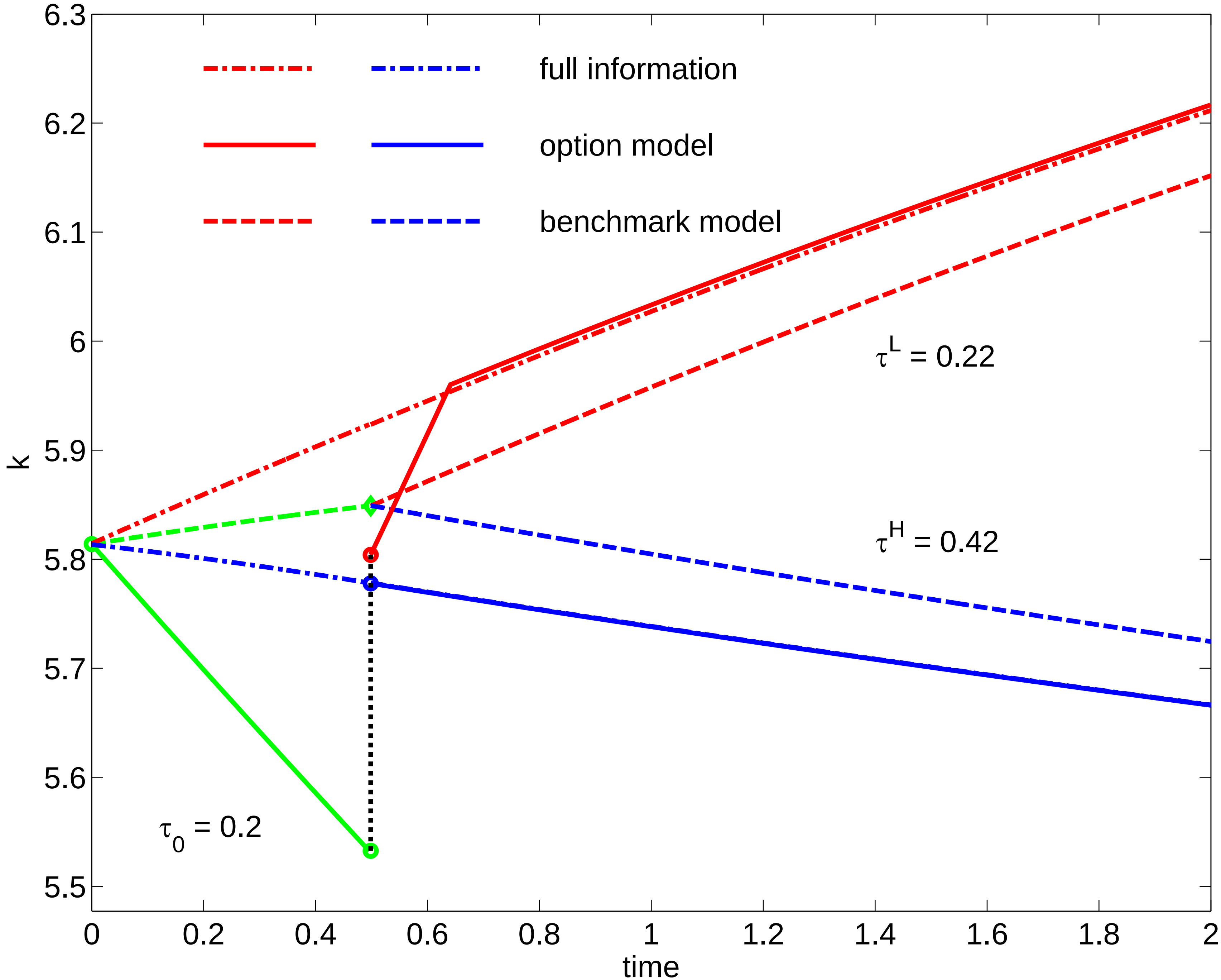


Figure 3: capital stock



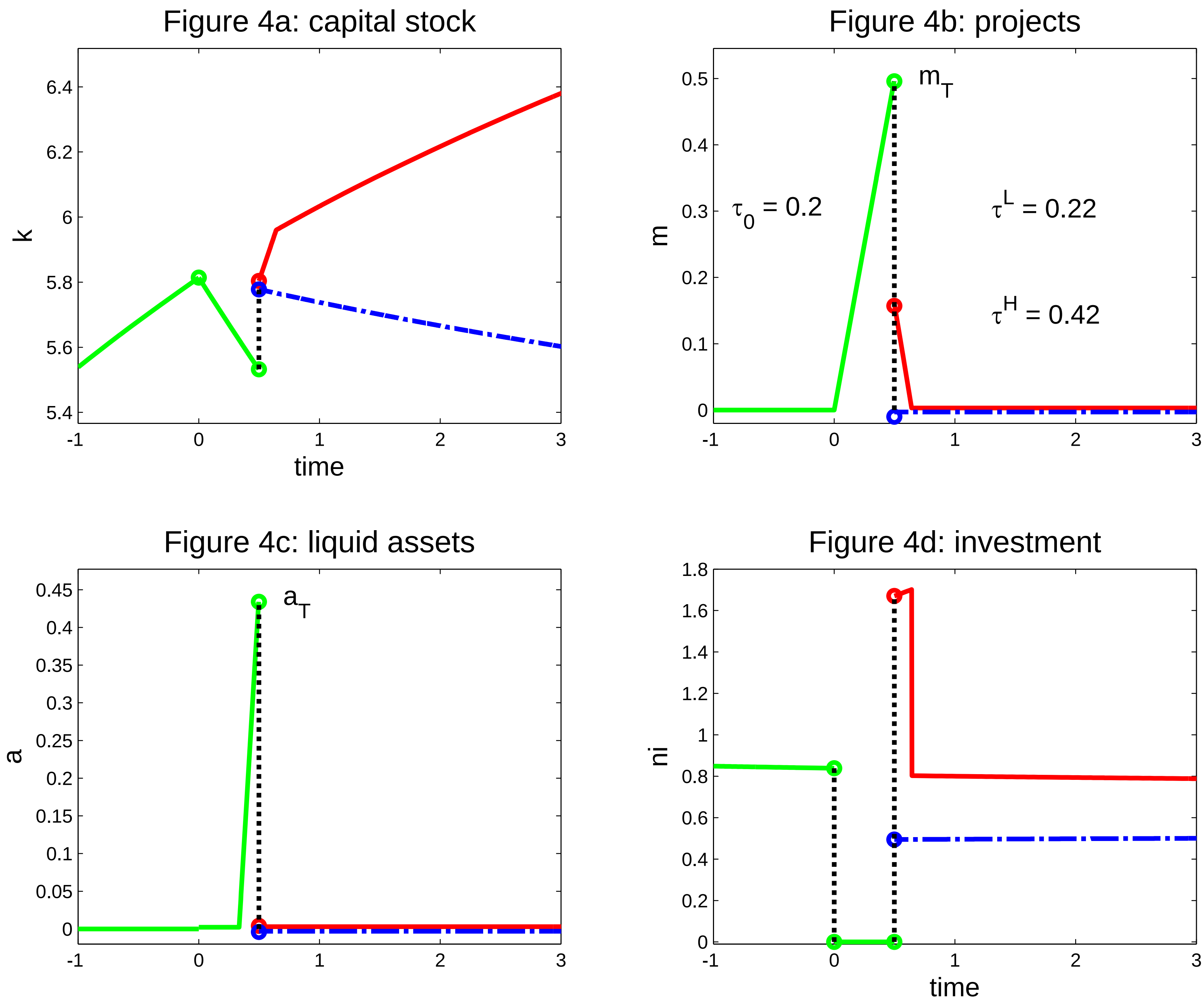


Figure 4a: capital stock
k
time
Figure 4b: projects
m
m_T
$\tau_0 = 0.2$
$\tau^L = 0.22$
$\tau^H = 0.42$
Figure 4c: liquid assets
a
a_T
Figure 4d: investment
ni
time

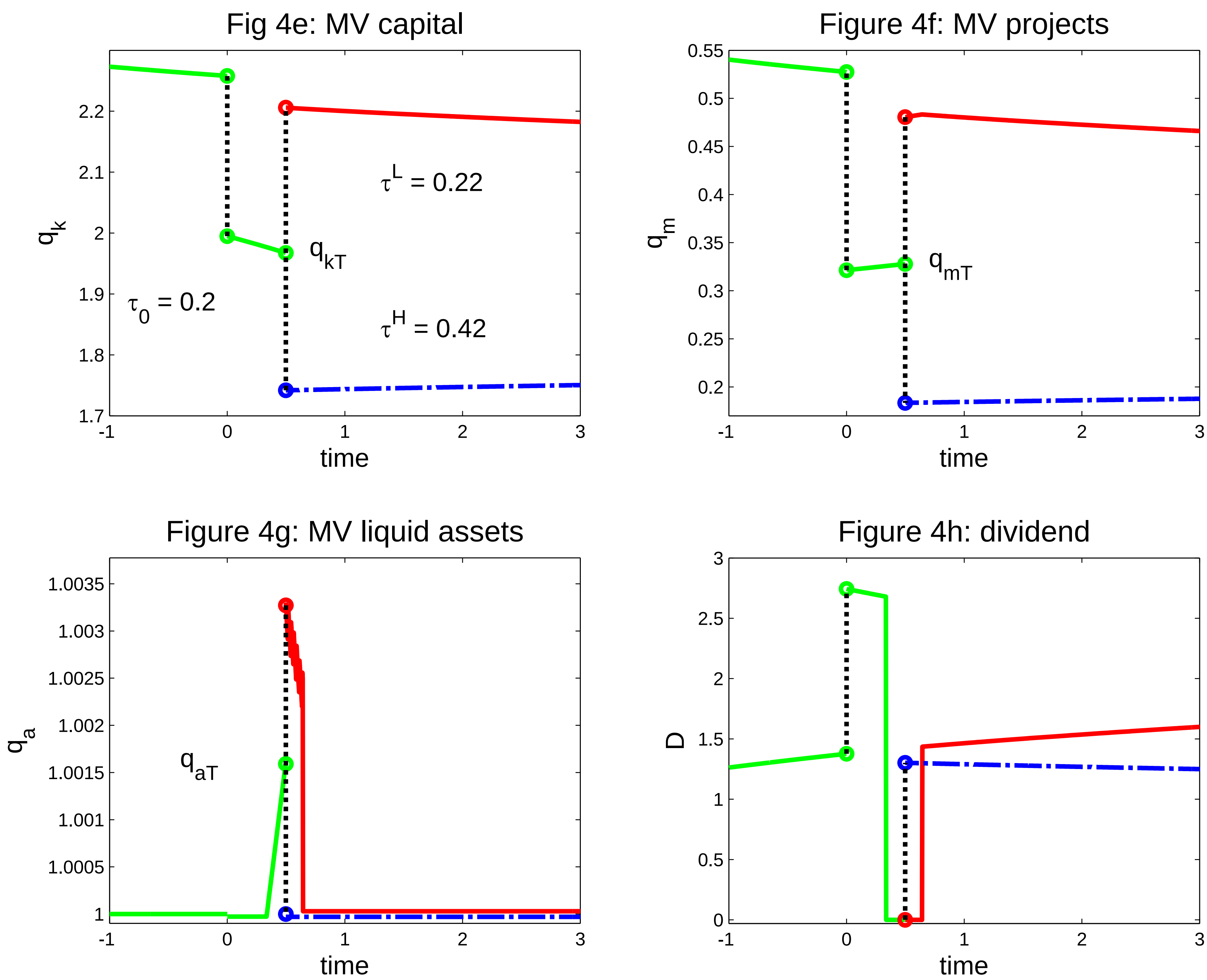


Fig 4e: MV capital
Figure 4f: MV projects
Figure 4g: MV liquid assets
Figure 4h: dividend
q_k
q_m
q_a
D
time
$\tau^L = 0.22$
$\tau_0 = 0.2$
$\tau^H = 0.42$
q_{kT}
q_{mT}
q_{aT}

Figure 5a: capital stock

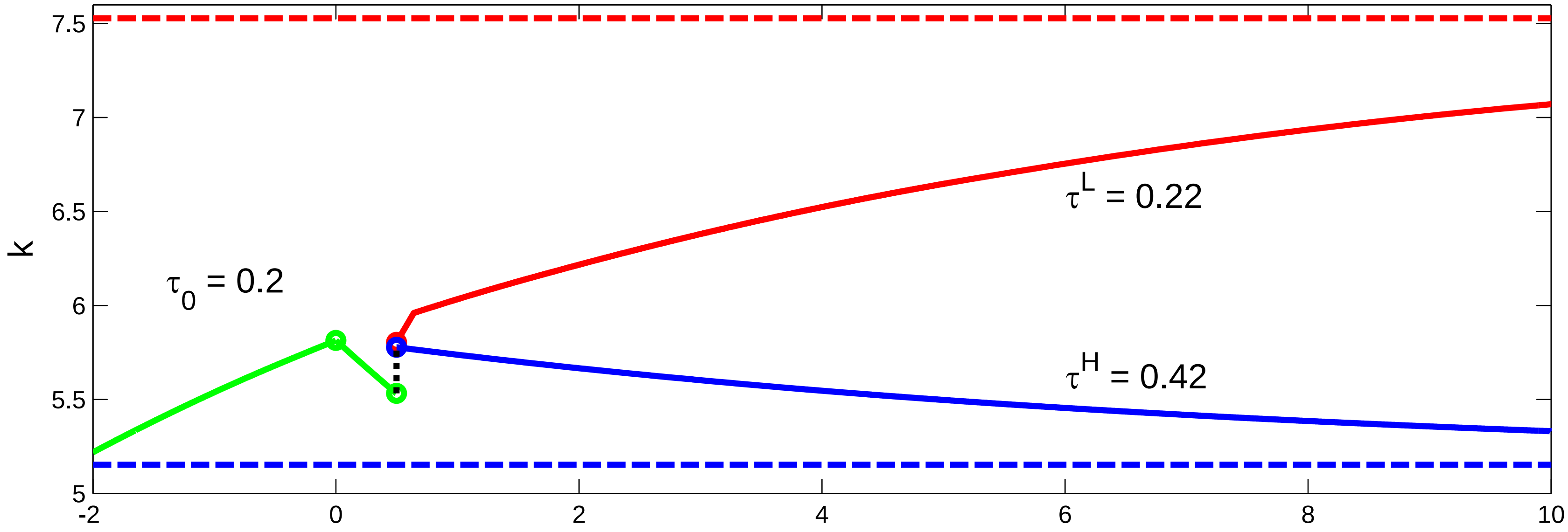


Figure 5b: investment