Recursive valuation and sentiments

Lars Peter Hansen

Bendheim Lectures, Princeton University

Recursive valuation and sentiments

Abstract

Expectations and uncertainty about growth rates that might prevail in the future influence the valuation of risky claims to consumption. I explore this mechanism using the recursive utility model pioneered by Koopmans, Kreps and Porteus, Epstein and Zin and others. This model gives a structured way to investigate how beliefs about the future are reflected in current-period assessments, including in the continuation values of prospective consumption processes and in the stochastic discount factors used to represent prices over alternative investment horizons. Thus the forward-looking nature of the recursive utility model provides an additional channel for which *sentiments* about the future matter. Using some recently developed methods for studying stochastic processes with uncertain growth, I provide revealing characterizations by exploring some limiting cases and suggesting alternative interpretations for sentiments.

Motivation

- Explore ways in which expectations and uncertainty about future growth rates influence risky claims to consumption.
- Use a recursive utility model pioneered by Koopmans, Kreps and Porteus and others that, by design, can make beliefs about uncertain events figure prominently in asset valuation.
- Provide novel and revealing characterizations that will help us understand better how this mechanism operates.

RECURSIVE PREFERENCES

Koopmans initiated an important line of research on recursive preferences that pushed beyond the additive discounted utility framework.

Some References:

- Stationary Ordinal Utility and Impatience - Koopmans, Econometrica 1960
- Koopmans, Diamond and Williamson - Econometrica 1964
- Kreps and Porteus - Econometrica 1978
- Epstein and Zin - Econometrica 1989

KOOPMANS AND RECURSIVE UTILITY

Utility representation:

$$V_t = \Phi[U(C_t), V_{t+1}]$$

as a generalization of

$$V_t = U(C_t) + \exp(-\delta) V_{t+1}$$

where C_t is the current period consumption vector, V_t is the "continuation value" or what Koopmans called the "prospective" utility.

Uncertainty

Kreps-Porteus representation

$$V_t = \Phi\left[U(C_t), E\left(V_{t+1}|\mathcal{F}_t\right)\right]$$

as a generalization of expected utility

$$V_t = U(C_t) + \exp(-\delta)E\left(V_{t+1}|\mathcal{F}_t\right).$$

K-P does not reduce intertemporal compound consumption lotteries. Intertemporal composition of risk matters.

I will feature a convenient special case

$$V_t = \left[(\zeta C_t)^{1-\rho} + \exp(-\delta)\left[\mathcal{R}_t(V_{t+1})\right]^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

where

$$\mathcal{R}_t\left(V_{t+1}\right) = \left(E\left[(V_{t+1})^{1-\gamma}|\mathcal{F}_t\right]\right)^{\frac{1}{1-\gamma}}$$

where $\frac{1}{\rho}$ is the elasticity of intertemporal substitution and γ is a risk aversion parameter. Epstein and Zin.

RECURSIVE UTILITY

- Continuation values

$$V_t = \left[(\zeta C_t)^{1-\rho} + \exp(-\delta)\left[\mathcal{R}_t(V_{t+1})\right]^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

  where

$$\mathcal{R}_t\left(V_{t+1}\right) = \left(E\left[(V_{t+1})^{1-\gamma}|\mathcal{F}_t\right]\right)^{\frac{1}{1-\gamma}}$$

  Induces some interesting nonlinearities in valuation.

- Intertemporal marginal rate of substitution

$$\frac{S_{t+1}}{S_t} = \exp(-\delta)\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\left[\frac{V_{t+1}}{\mathcal{R}_t(V_{t+1})}\right]^{\rho-\gamma}.$$

  Depends on continuation values, which gives a channel for sentiments to matter.

  Used to represent asset prices.

Talk outline

- Mathematical setup
- Two related applications
  - Continuation values for infinite-horizon problems
  - Asset pricing over alternative investment horizons
- Perron-Frobenius theory and martingales
- Applications revisited
- Estimated long-run risk model
- Robustness and beliefs

Approach

- Use Markov formulations and martingale methods to study compounding in stochastic environments
- Allow for nonlinear time series models including models of stochastic volatility and stochastic regime shifts.
- Use the long-term as a frame of reference.

Explore the implications of state dependent compounding when we alter the forecast or consumption horizon.

Study continuation values, asset values and growth.

Discrete-time formulation

- Markov process X.
- Additive functional

$$Y_t = \sum_{j=1}^{t} \kappa(X_j, X_{j-1})$$

- Multiplicative functional

$$M_t = \exp(Y_t) = \prod_{j=1}^{t} \exp[\kappa(X_j, X_{j-1})]$$

- The product of two multiplicative functions is a multiplicative functional.

Use multiplicative functionals to model state dependent growth and discounting.

EXAMPLE: MIXTURE OF NORMALS MODEL

- Let $\{W_{t+1}\}$ be a multivariate iid sequence of standard normals.
- Construct an additive functional:

$$Y_{t+1} - Y_t = Z_t \cdot \mu + Z_t \cdot \Lambda W_{t+1}$$

  where Z is a component of X and evolves as a finite state Markov chain $Y_0 = 0$.
- Construct the multiplicative functional:

$$M_t = \exp(Y_t)$$

  and study conditional expectations. Recursive structure to the compounding over time.

UTILITY RECURSION RECONSIDERED

- Recall

$$V_t = \left[(\zeta C_t)^{1-\rho} + \exp(-\delta) \left[\mathcal{R}_t(V_{t+1}) \right]^{1-\rho} \right]^{\frac{1}{1-\rho}} .$$

where

$$\mathcal{R}_t \left(V_{t+1} \right) = \left(E \left[(V_{t+1})^{1-\gamma} | \mathcal{F}_t \right] \right)^{\frac{1}{1-\gamma}} .$$

- To adjust for growth, exploit homogeneity and divide by C_t to obtain:

$$\frac{V_t}{C_t} = \left[\zeta^{1-\rho} + \exp(-\delta) \left[\mathcal{R}_t \left(\frac{V_{t+1}}{C_{t+1}} \frac{C_{t+1}}{C_t} \right) \right]^{1-\rho} \right]^{\frac{1}{1-\rho}} ,$$

When consumption is a multiplicative functional, $\frac{V_t}{C_t}$ will be a function of the Markov state X_t.

RESTATING THE RECURSION

- Explore the parameter region $\gamma > \rho$ and define

$$1 - \nu = \frac{1 - \gamma}{1 - \rho}$$

Restrict $\rho \neq 1$ in what follows. ($\rho = 1$ requires a separate argument.)

- Consumption dynamics

$$\log C_{t+1} - \log C_t = \kappa(X_{t+1}, X_t),$$

and solution form:

$$\left(\frac{V_t}{C_t}\right)^{1-\rho} = h(X_t).$$

- Formalism:

$$\mathbb{V}f(x) = E\left(\exp[(1-\gamma)\kappa(X_{t+1}, X_t)]f(X_{t+1}) | X_t = x\right),$$

Restated recursion

$$h(x) = \zeta^{1-\rho} + \exp(-\delta)\left[\mathbb{V}h^{1-\nu}(x)\right]^{\frac{1}{1-\nu}}$$

Asset pricing over alternative investment horizons

Multi-period return:

$$\frac{G_t}{E\left(S_t G_t | X_0 = x\right)}$$

Logarithm of the expected return adjusted for horizon:

$$\frac{1}{t}\log E\left(G_t | X_0 = x\right) - \frac{1}{t}\log E\left(S_t G_t | X_0 = x\right)$$

- In valuation problems there are two forces at work - stochastic growth G and stochastic discounting S. Study product SG.
- Term structure of risk and shock prices - look at value implications of marginal changes in growth exposure as represented by changes in G.
- Build recursions that exploit the Markov structure.

Warmup

- Normal mixture model:

$$Y_{t+1} - Y_t = Z_t \cdot \mu + Z_t \cdot \Lambda W_{t+1}$$

  where Z evolves a finite state Markov chain. Realized value of Z_t is a coordinate vector.
- Let $f(X_{t+1}) = \mathbf{g} \cdot Z_t$:

$$\begin{aligned} z'\mathbb{M}\mathbf{g} &= E\left[\frac{M_{t+1}}{M_t} f(X_{t+1}) | X_t = x\right] \\ &= E\left[\exp(Y_{t+1} - Y_t)\mathbf{g} \cdot Z_{t+1} | Z_t = z\right] \end{aligned}$$

  where $\mathbb{M}$ is a matrix with nonnegative entries.
- Characterize $\mathbb{M}^j$. Grows or decays geometrically with the rate given by logarithm of the dominant eigenvalue.
- Perron-Frobenius theory - dominant eigenvalue has an eigenvector with positive entries.
- Adjust for growth or decay using the eigenvalue; construct a new probability matrix using the eigenvector.

Perron-Frobenius Theory/ Martingales

- Solve,

$$E\left[M_1 e(X_1) | X_0 = x\right] = \exp(\eta) e(x)$$

  where e is strictly positive. Eigenvalue problem.

- Construct martingale

$$\hat{M}_t = \exp(-\eta t) M_t \left[\frac{e(X_t)}{e(X_0)}\right].$$

- Invert to obtain factorization

$$M_t = \exp(\eta t) \hat{M}_t \left[\frac{e(X_0)}{e(X_t)}\right].$$

$\exp(\eta t)$ is the eigenvalue for horizon t and e the eigenfunction.

Multiplicative martingales

Factorization:

$$M_t = \exp(\eta t)\hat{M}_t \left[\frac{e(X_0)}{e(X_t)}\right].$$

Change of probability measure:

$$\hat{E}\left[f(X_t)|X_0 = x\right] = E\left(\hat{M}_t f(X_t)|X_0 = x\right)$$

- preserves Markov structure
- at most one is stochastically stable - Hansen-Scheinkman

STOCHASTIC STABILITY

$$\exp(-\eta t) E\left[M_t f(X_t) | X_0 = x\right] = e(x) \hat{E}\left[\frac{f(X_t)}{e(X_t)} | X_0 = x\right]$$

Under stochastic stability and the moment restriction:

$$\hat{E}\left[\frac{f(X_t)}{e(X_t)}\right] < \infty,$$

the right-hand side converges to:

$$e(x) \hat{E}\left[\frac{f(X_t)}{e(X_t)}\right].$$

Common state dependence independent of f.

Change of measure revisited

Recall,

$$M_t = \exp(\eta t)\hat{M}_t \left[\frac{e(X_0)}{e(X_t)}\right].$$

Consider the stochastic discount factor $S = M$.

Alternative factorization - risk neutral dynamics.

- Risk-neutral adjustment is a local or one-period adjustment whereas our adjustment features the long-term valuation.
- Short-term interest rates are typically state dependent whereas η is not. The state dependence of short-term interest rates adjusts for risks over multi-period horizons.
- Our change of measure features risk adjustments over multiple investment horizons with direct characterizations of limiting behavior.

We are interested in other applications as well that include adjustments for stochastic growth.

Continuation-value recursion

- Define

$$1-\nu=\frac{1-\gamma}{1-\rho}$$

where $\nu>0$. We presume that $\rho\neq 1$ in what follows.

- Consumption dynamics

$$\log C_{t+1}-\log C_t=\kappa(X_{t+1},X_t),$$

Solution of the form:

$$\left(\frac{V_t}{C_t}\right)^{1-\rho}=h(X_t).$$

- Formalism:

$$\mathbb{V}f(x)=E\left(\exp[(1-\gamma)\kappa(X_{t+1},X_t)]f(X_{t+1})|X_t=x\right),$$

Restated recursion

$$h(x)=\zeta^{1-\rho}+\exp(-\delta)\left[\mathbb{V}h^{1-\nu}(x)\right]^{\frac{1}{1-\nu}}$$

Iterating on the recursion

- Construct

  $$\log M_{t+1} - \log M_t = (1-\gamma)\kappa(X_{t+1}, X_t) = (1-\gamma)(\log C_{t+1} - \log C_t)$$

  which depends only on γ and not ρ.

- Factor

  $$M_t = \exp(\eta t)\hat{M}_t \left[\frac{e(X_0)}{e(X_t)}\right].$$

- Use change of measure:

  $$\begin{aligned} \mathbb{V}f(x) &= E\left(\exp[(1-\gamma)\kappa(X_{t+1}, X_t)]f(X_{t+1})|X_t = x\right), \\ &= \exp(\eta)\hat{E}\left[\frac{f(X_{t+1})}{e(X_{t+1})}|X_t = x\right] e(x) \end{aligned}$$

BOUNDS

- Recall

$$h(x) = \zeta^{1-\rho} + \exp(-\delta)\left[\mathbb{V}h^{1-\nu}(x)\right]^{\frac{1}{1-\nu}}$$

- The parameter restriction that is a necessary condition for finite values for an infinite horizon:

$$\delta \geq \eta\frac{1-\rho}{1-\gamma}$$

- Moment inequality

$$\hat{E}\left[e(X_t)^{\frac{1}{\nu-1}}\right] < \infty$$

implies a bound on the infinite-horizon continuation value. Deduced by applying Jensen's Inequality to

$$\hat{E}\left[f(X_{t+1})^{1-\nu}|X_t = x\right]$$

for an appropriately defined f.

In parametric examples these inequalities imply parameter restrictions that are typically ignored in practice.

What happens as we approach the discount factor limit?

Recursive utility mechanism features beliefs about the future or "sentiments".

Makes these as potent as possible.

- Recall

$$h(x) = \zeta^{1-\rho} + \exp(-\delta) \left[\mathbb{V} h^{1-\nu}(x) \right]^{\frac{1}{1-\nu}}$$

- Drive ζ to zero (scale doesn't matter) and δ to its bound. Then the equation simplifies to

$$h^{1-\nu}(x) = \exp(-\eta) \mathbb{V} h^{1-\nu}(x)$$

- Continuation value function (relative to consumption) converges to

$$e(x)^{\frac{1}{1-\gamma}}$$

up to scale.

Limiting representation of the stochastic discount factor

$$\begin{aligned}
\frac{S_{t+1}}{S_t} &= \exp(-\delta)\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\left[\frac{V_{t+1}}{\mathcal{R}_t(V_{t+1})}\right]^{\rho-\gamma} \quad \text{in general.}\\
\frac{S_{t+1}}{S_t} &= \exp(-\eta)\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left[\frac{e(X_{t+1})}{e(X_t)}\right]^{\frac{\rho-\gamma}{1-\gamma}} \quad \text{in the limit.}
\end{aligned}$$

- Stochastic discount factors for power utility (constructed with γ) and recursive utility share the same martingale components.
- Changing ρ does not alter the martingale component.

Estimated example

Motivated by the work of Bansal and Yaron, I fit the following model to the aggregate time series data and specified in continuous time.

$$
\begin{aligned}
dX_t^{[1]} &= A_{11} X_t^{[1]} dt + \sqrt{X_t^{[2]}} B_1 dW_t, \\
dX_t^{[2]} &= A_{22} (X_t^{[2]} - 1) dt + \sqrt{X_t^{[2]}} B_2 dW_t \\
dY_t &= \mu dt + H_1 X_t^{[1]} dt + \sqrt{X_t^{[2]}} F dW_t.
\end{aligned}
$$

The continuous time autogregressive coefficients are:

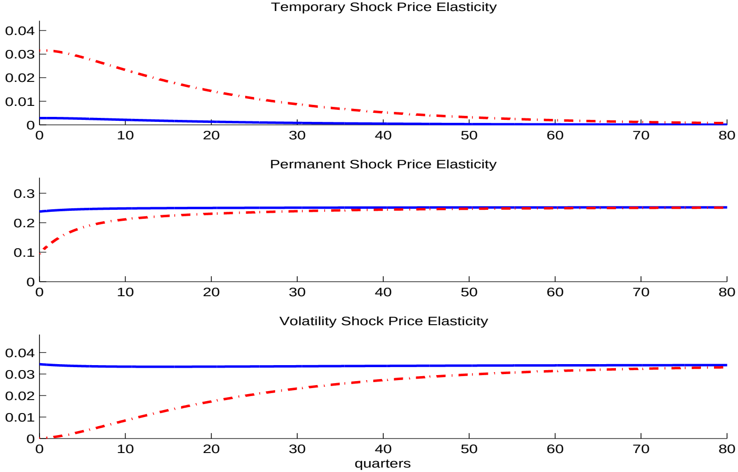
$$
A_{11} = \begin{bmatrix} -.05 & .01 \\ 0 & -.29 \end{bmatrix} \quad A_{22} = -.07.
$$

The shock loadings for the components of the state vector are:

$$
B_1 = \begin{bmatrix} .047 & .018 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 0 & -.18 \end{bmatrix}
$$

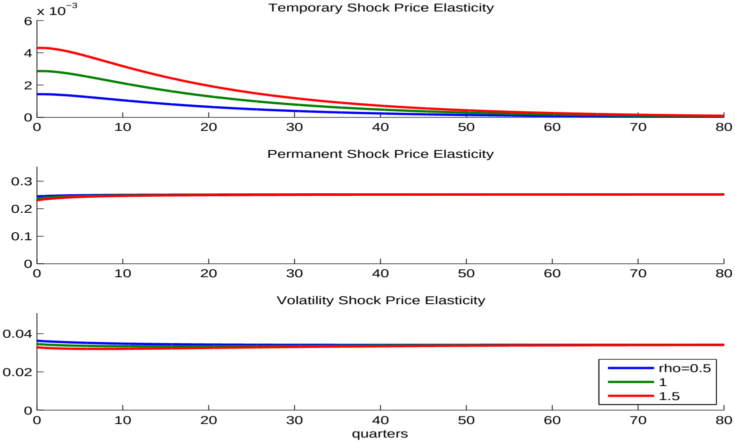
$$
F = \begin{bmatrix} -.0011 & .0044 & 0 \end{bmatrix} \quad H_1 = \begin{bmatrix} .0021 & .0014 \end{bmatrix}
$$

Shock-price trajectories for power and recursive utility



See second lecture for interpretation and construction details.

SHOCK-PRICE TRAJECTORIES FOR ALTERNATIVE VALUES OF THE EIS



See second lecture for interpretation and construction details.

Impact of stochastic volatility

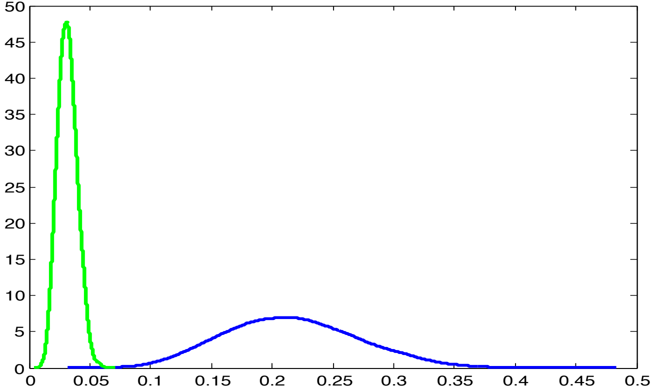


Figure: Densities for shock price elasticities for exposure to the growth rate shock. Green volatility shock and blue growth shock.

IMPACT OF PARAMETER ESTIMATION

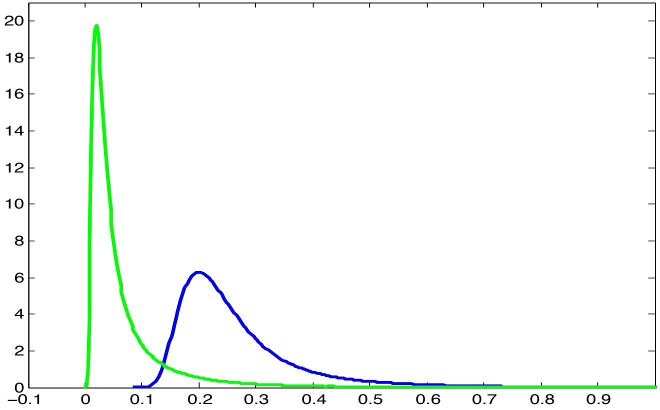


FIGURE: Densities for shock price elasticities for exposure to the growth rate shock. Green volatility shock an blue growth shock.

See also Hansen, Heaton and Li (JPE).

Robustness and distorted beliefs

- Lack of investor confidence in the models they use.
- Investors explore alternative specifications for probability laws subject to penalization.
- Martingale is the implied "worst case" model. Parameter $\theta > 0$ determines the magnitude of the penalization where $\gamma > 1$ and

  $$\theta = \left| \frac{1-\rho}{1-\gamma} \right| .$$

  The parameters are now ρ and θ where ρ continues to measure the elasticity of intertemporal substitution.
- Related methods have a long history in "robust" control theory and statistics.
- Axiomatic treatments in recent decision theory papers.

Robustness and distorted beliefs

- In empirical applications it is common to assume a large value of the risk aversion parameter γ.
- Instead appeal to a concern about robustness.
- For $v_{t+1} = |1-\rho| \log V_{t+1}$ and solve

$$\min_{m \geq 0, E[m|\mathcal{F}_t]=1} E\left[v_{t+1}|\mathcal{F}_t\right] + \frac{1}{\theta} E\left[m \log m|\mathcal{F}_t\right]$$
$$= -\theta \log E\left[\exp\left(-\frac{1}{\theta} v_{t+1}\right)|\mathcal{F}_t\right].$$

- Minimizing m:

$$m_{t+1} = \frac{\exp\left(-\frac{1}{\theta} v_{t+1}\right)}{E\left[\exp\left(-\frac{1}{\theta} v_{t+1}\right)|\mathcal{F}_t\right]} = \frac{(V_{t+1})^{1-\gamma}}{E\left[(V_{t+1})^{1-\gamma}|\mathcal{F}_t\right]}.$$

This gives rise to the exponential tilting solution as m tilts the density in directions that have the largest adverse consequences for the continuation value. Altered beliefs.

See third lecture for more details.

Where does this leave us?

- The flat term structure for recursive utility shows the potential importance of macro growth components on asset pricing.
- Typical rational expectations modeling assumes investor confidence and uses the "cross equation" restrictions to identify long-term growth components from asset prices. Instead do asset prices identify "subjective beliefs" of investors and risk aversion?
- Predictable components of macroeconomic growth and volatility are hard for an econometrician to measure from macroeconomic data.

Questions

- What are the interesting shocks?
- Where does investor confidence come from when confronted by weak sample evidence? Motivates my interest in modeling investors who have a concern for model specification.
- What about learning? Concerns about model specification of the type I described abstract from learning.