These notes derive "Ramsey's optimal tax formula" that we discussed in lecture. Econ 203 students should understand the logic behind the formula and how the formula might be applied to government policies, but it is *not* expected that Econ 203 students would be able to reproduce this algebra on an exam or problem set.

I.   Deadweight Loss of a Tax

Consider a tax of t_i *per unit* of good i.

<u>Assumption 1</u> Hicksian demand curves are linear in the relevant range. For good i, the demand curve is:

$$p_i = a_i - b_i x_i$$

where x_i is the quantity of i consumed, p_i is the price paid by consumers, and a_i and b_i are constants.

If the taxes we study are sufficiently small, Assumption 1 might be justified as a Taylor approximation to a nonlinear demand curve.

<u>Assumption 2</u> The supply of good i is infinitely elastic at price $= c_i$.[1]

You may remember from microeconomics that the dead-weight loss of a tax is the area between the supply and demand curves and to the right of the quantity demanded x_i' when the tax is in place. Employing Assumptions 1 & 2, Figure 1 draws the Hicksian demand curve as a downward sloping line with intercept a_i and the supply curve as a horizontal line with intercept c_i.

With taxes, the equilibrium price of good i is $p_i = c_i$ - the same for producers and consumers - and the quantity traded in the market is x_i^*. With a tax per unit of t_i, the equilibrium price of good i is $p_i = c_i + t_i$ and the quantity of good i is x_i'.

Because we have drawn a Hicksian demand curve, the two shaded areas are what the consumer is willing to pay to avoid the tax.[2] The entire shaded area is not a loss to "society," however, because the government does get some revenue from the tax. The revenue is the tax per unit times the number of units sold, $T_i = t_i x_i'$, and can be shown graphically as the shaded box in the figure. The net loss - or "deadweight loss" - from the tax on good i is therefore the shaded triangle labeled

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[1]For a derivation with imperfectly elastic supply curves, see Ramsey (1927).

[2]You may remember from microeconomics that the area under the Hicksian demand curve is the "expenditure function" - the amount of money a consumer must spend to attain a given level of utility. With the higher price $c_i + t_i$, the area under the demand curve increases by exactly the shaded area which is the extra money the consumer needs to maintain his utility.

DWL$_i$.
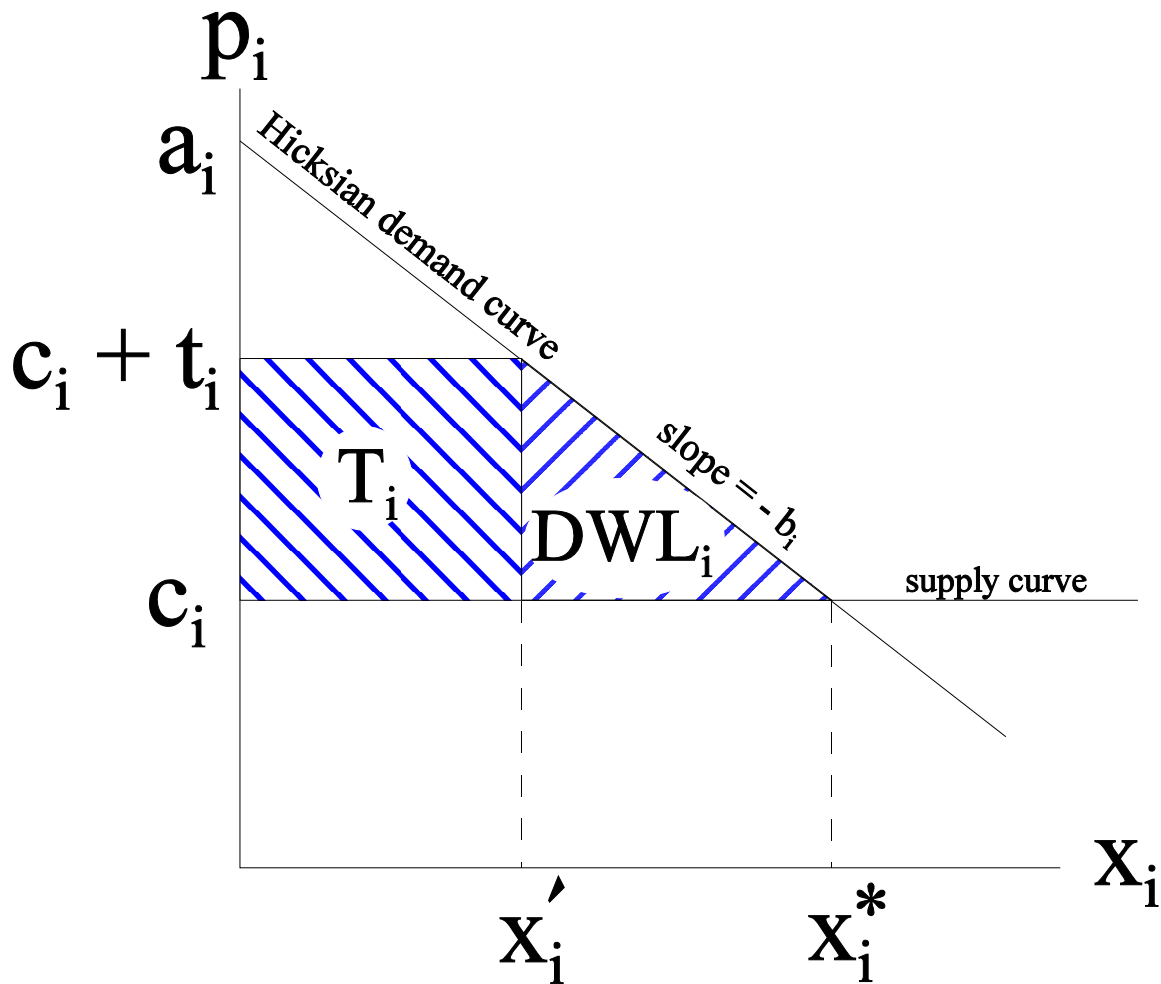
Figure 1 Geometric Analysis of the Deadweight Loss of a Tax

The formula for the good i demand curve is $p_i = a_i - b_i x_i$ or, equivalently, $x_i = (a_i - p_i)/b_i$. Since we have a formula for the demand curve, we can compute the change in demand $(x_i^* - x_i')$ as a result of the tax.

$$x_i^* - x_i' = \frac{a_i - c_i}{b_i} - \frac{a_i - c_i - t_i}{b_i} = \frac{t_i}{b_i}$$

Definition 1   Define the *ad valorem tax rate* of good *i* to be taxes paid per unit of sales of good *i* and denote that tax rate τ_i.

$$\tau_i \equiv \frac{\text{tax paid on good i}}{\text{pre-tax sales of good i}} = \frac{t_i x_i}{c_i x_i} = \frac{t_i}{c_i}$$

Notice that ad valorem tax rates are easily compared across goods. An ad valorem tax rate of 0.10 is a 10% tax on the sales of a good. "Per unit" taxes are tougher to compare across goods. For example, a tax of $0.10 per unit is a very significant tax on a can of soda but a very insignificant tax on an automobile.

Definition 2   Define the *magnitude of the demand elasticity* of good i to be the percentage reduction in quantity demanded for every percentage change in the price of good i evaluated at $p_i = c_i$ and denote that elasticity η_i.

$$\eta_i \equiv \left. -\frac{dx_i}{dp_i}\frac{p_i}{x_i}\right|_{t_i = 0}$$

$$= \frac{c_i}{a_i - c_i}$$

The second equality follows from our formula for the demand function $x_i = (a_i - p_i)/b_i$. Using that formula for the demand function and the geometry displayed in Figure 1, we can compute a formula for the deadweight loss from a tax on good i:

$$DWL_i \equiv \frac{1}{2}(base)(height)$$

$$\equiv \frac{1}{2}\left(\frac{t_i}{b_i}\right)(t_i) \tag{*}$$

$$\equiv \frac{1}{2}\left(\frac{\tau_i}{b_i}c_i\right)(\tau_i c_i) = \frac{1}{2}\frac{c_i^2}{b_i}\tau_i^2$$

Notice that deadweight losses are in dollars: a tax per unit of a good (t_i) times a quantity of goods (t_i/b_i).

II. Ramsey's Problem

Ramsey asked the following two dual questions:

* "For a given amount of revenue to be extracted from a consumer, what tax system makes the consumer happiest?"

• "For a given level of happiness, how can we extract the most revenue from a consumer?"

These are dual questions just like the classic consumer problems in price theory "What choices maximize utility subject to the budget constraint?" and its dual "What choices minimize expenditure subject to a utility constraint?" This note sets up the first problem, but keep in mind that the second problem is its dual.

The first problem can be expressed mathematically as:

$$\min_{\tau_1, \tau_2, \dots, \tau_N} \sum_{i=1}^{N} DWL_i \quad \text{s.t.} \quad \sum_{i=1}^{N} T_i \geq G$$

The choice variables are the tax rates on the N different goods. The objective is the total deadweight loss from all of the taxes. Notice that summing deadweight losses across goods makes some economic sense because deadweight loss is measured in dollars. As long as we are only thinking about the taxes to be levied on a *single consumer*, summing dollars also makes sense from a "social welfare" point of view. If instead one consumer consumes one set of goods and another consumer consumers a different set of goods, then a dollar of deadweight loss on one good may not be "socially" equal to a dollar of deadweight loss on another good.[3] T_i is the tax revenue collected from taxes on good i, so the constraint requires that total tax revenue be at least the required government spending G.

To make the problem interesting and more tractable, three assumptions are useful:

<u>Assumption 3</u> Lump sum taxes are not available. In addition to the N goods under consideration, there is at least one other good that cannot be taxed.

<u>Assumption 4</u> Hicksian cross-price elasticities are zero.

<u>Assumption 5</u> The required government spending G is fixed - it does not depend on the chosen mix of taxes.

Assumption 3 makes Ramsey's problem interesting. If lump sum taxes were available, using those and setting all ad valorem tax rates to zero would generate zero deadweight loss and maximize utility. If there were no untaxed good, making all tax rates equal would not distort any relative prices and therefore not create any deadweight loss.

Assumption 4 makes the problem easier - we don't have to worry about the possibility that a tax on good i might affect the demand for good j. Notice that formula for the demand function shown in

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[3]For example, it may be "socially" desirable to tax the rich more heavily than the poor.

Assumption 1 satisfies Assumption 4. Assumption 5 also makes the problem easier, although see Mulligan (1996), Becker and Mulligan (1998), or our lecture notes on "How the Economy Influences Policy" for an argument that the answer to Ramsey's problem with Assumption 5 should not guide tax policy.

With the Assumptions 3-5, the LaGrangian for Ramsey's problem is:

$$\mathcal{L} \equiv \sum_{i=1}^{N} \frac{1}{2} \frac{1}{b_i} \tau_i^2 c_i^2 + \lambda \left[G - \sum_{i=1}^{N} \tau_i c_i \frac{a_i - c_i - \tau_i c_i}{b_i} \right]$$

The first sum in the LaGrangian is the sum of deadweight loss for each good and uses the formula (*) for the deadweight loss derived in Section I. The sum in the brackets is the sum of tax revenues from each good and uses the formula $T_i = t_i x_i' = \tau_i c_i (a_i - p_i)/b_i$. λ is the LaGrange multiplier on the government's budget constraint and therefore might be thought of as the "marginal deadweight loss of government spending."

Taking the first order condition with respect to τ_i and solving for τ_i, we get:[4]

$$\frac{\tau_i c_i^2}{b_i} = \lambda c_i \frac{a_i - c_i}{b_i} - 2\lambda \frac{\tau_i c_i^2}{b_i}$$

$$\Downarrow$$

$$\tau_i c_i \frac{c_i}{a_i - c_i} = \lambda c_i - 2\lambda \tau_i c_i \frac{c_i}{a_i - c_i}$$

$$\Downarrow$$

$$\tau_i = \frac{\lambda}{1 + 2\lambda} \frac{a_i - c_i}{c_i}$$

$$\Downarrow$$

$$\tau_i = \frac{\lambda}{1 + 2\lambda} \frac{1}{\eta_i} \tag{**}$$

The first equality equates the marginal deadweight loss to the marginal revenue of a change in good i's tax rate. The equation depend only on the tax, supply, and demand parameters of good i and not those of other goods because we have assumed that the Hicksian cross price elasticities are zero. The first term on the RHS is proportional to the quantity of i demanded at $p_i = c_i$. The second term

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[4]You can check that, given our assumptions, the second order conditions of the problem hold.

reflects the tax revenue lost because a tax increase decreases the demand for good i. Without the second term, we would only have the first term - a change in the tax rate would raise revenue according to the quantity of x demanded. As one might expect, the second term is important relative to the first when the elasticity of demand is of greater magnitude. This fact can be seen in the second equality (which multiplies the first by $b_i/(a_i\text{-}c_i)$) by noting that $c_i/(a_i\text{-}c_i)$ is the magnitude of the demand elasticity. The second equality also emphasizes that term on the LHS - the marginal deadweight loss of the tax on good i - is relatively important when the magnitude of the demand elasticity is large. An elastically demanded good therefore has a high marginal deadweight loss (the LHS term) and is a poor source of revenue (the second term on the RHS), suggesting that it is not optimal to tax an elastically demanded good heavily.

The final equality derives from the definition of η_i, the magnitude of demand elasticity of good i. I refer to (**) as "Ramsey's Optimal Tax Formula."

III. Properties of Ramsey's Solution

There are three properties of Ramsey's solution worth remembering.

Property 1   Any good that can be taxed should be taxed to some degree.

Property 2   Goods with higher demand elasticities should be taxed less.

Property 3   The size of government affects all tax rates proportionally.

Property 1 follows from Ramsey's formula, which requires $\tau_i > 0$ for all i in the maximization problem. Properties 2 and 3 can be proved by dividing the Ramsey tax formulas for any two goods *i* and *j*:

$$\frac{\tau_i}{\tau_j} = \frac{\eta_j}{\eta_i}$$

A change in the size of government does not affect the optimal ratio of the two tax rates. In terms of Ramsey's optimal tax formula, think of *G* as affecting the formula through λ. The more revenue that must be raised, the greater the marginal deadweight loss of government spending. According to (**) the marginal deadweight loss of government spending affects all taxes proportionally.

IV.   References

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