

GAME THEORY I

A STRATEGIC SITUATION (DUE TO BEN POLAK)

		Player 2	
		α	β
Player 1	α	B-, B-	A, C
	β	C, A	A-, A-

SELFISH STUDENTS

		Selfish 2	
		α	β
Selfish 1	α	1, 1	3, 0
	β	0, 3	2, 2

SELFISH STUDENTS

		Selfish 2	
		α	β
Selfish 1	α	1, 1	3, 0
	β	0, 3	2, 2

- ▶ No matter what Selfish 2 does, Selfish 1 wants to choose α (and vice versa)

SELFISH STUDENTS

		Selfish 2	
		α	β
Selfish 1	α	1, 1	3, 0
	β	0, 3	2, 2

- ▶ No matter what Selfish 2 does, Selfish 1 wants to choose α (and vice versa)
- ▶ (α, α) is a sensible prediction for what will happen

NICE STUDENTS

		Nice 2	
		α	β
Nice 1	α	2, 2	1, 0
	β	0, 1	3, 3

NICE STUDENTS

		Nice 2	
		α	β
Nice 1	α	2, 2	1, 0
	β	0, 1	3, 3

- ▶ Each nice student wants to match the behavior of the other nice student

NICE STUDENTS

		Nice 2	
		α	β
Nice 1	α	2, 2	1, 0
	β	0, 1	3, 3

- ▶ Each nice student wants to match the behavior of the other nice student
- ▶ (α, α) or (β, β) seem sensible.

NICE STUDENTS

		Nice 2	
		α	β
Nice 1	α	2, 2	1, 0
	β	0, 1	3, 3

- ▶ Each nice student wants to match the behavior of the other nice student
- ▶ (α, α) or (β, β) seem sensible.
- ▶ We need to know what people think about each other's behavior to have a prediction

SELFISH VS. NICE

		Nice	
		α	β
Selfish	α	1, 2	3, 0
	β	0, 1	2, 3

SELFISH VS. NICE

		Nice	
		α	β
Selfish	α	1, 2	3, 0
	β	0, 1	2, 3

- ▶ Nice wants to match what Selfish does

SELFISH VS. NICE

		Nice	
		α	β
Selfish	α	1, 2	3, 0
	β	0, 1	2, 3

- ▶ Nice wants to match what Selfish does
- ▶ No matter what Nice does, Selfish wants to player α

SELFISH VS. NICE

		Nice	
		α	β
Selfish	α	1, 2	3, 0
	β	0, 1	2, 3

- ▶ Nice wants to match what Selfish does
- ▶ No matter what Nice does, Selfish wants to play α
- ▶ If Nice can think one step about Selfish, she should realize she should play α

SELFISH VS. NICE

		Nice	
		α	β
Selfish	α	1, 2	3, 0
	β	0, 1	2, 3

- ▶ Nice wants to match what Selfish does
- ▶ No matter what Nice does, Selfish wants to player α
- ▶ If Nice can think one step about Selfish, she should realize she should play α
- ▶ (α, α) seems the sensible prediction

OUTLINE

STRATEGIC FORM GAMES

SOLVING A GAME: NASH EQUILIBRIUM

COMPONENTS OF A GAME

Players: Who is involved?

Strategies: What can they do?

Payoffs: What do they want?

CHICKEN

		Player 2	
		Straight	Swerve
Player 1	Straight	0, 0	3, 1
	Swerve	1, 3	2, 2

CHOOSING A RESTAURANT

		Rebecca	
		P	V
Ethan	P	4, 3	1, 1
	V	0, 0	3, 4

WORKING IN A TEAM

2 players

Player i chooses effort $s_i \geq 0$

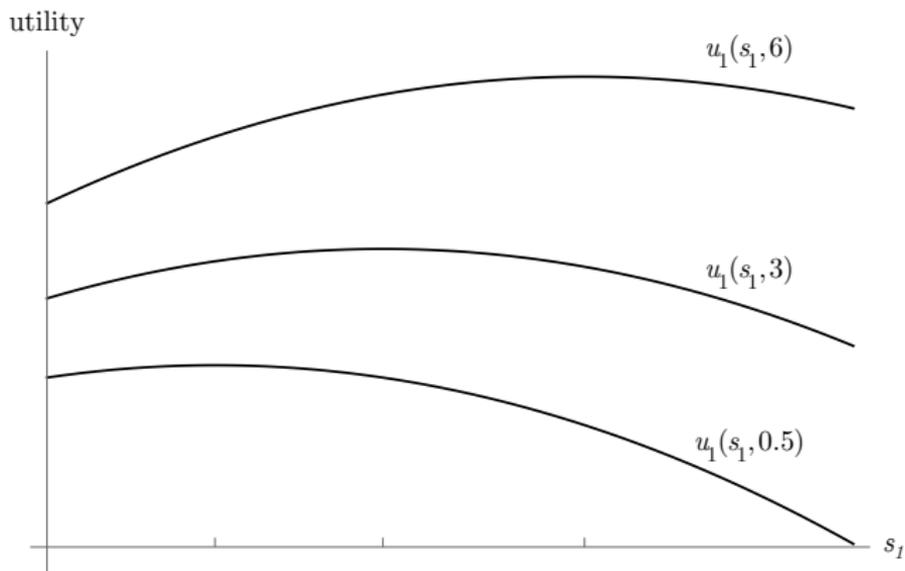
Jointly produce a product. Each enjoys an amount

$$\pi(s_1, s_2) = s_1 + s_2 + \frac{s_1 \times s_2}{2}$$

Cost of effort is s_i^2

$$u_i(s_1, s_2) = \pi(s_1, s_2) - s_i^2$$

PLAYER 1'S PAYOFFS AS A FUNCTION OF EACH PLAYER'S STRATEGY



CHOOSING A NUMBER

N players

Each player “bids” a real number in $[0, 10]$

If the bids sum to 10 or less, each player’s payoff is her bid

Otherwise players’ payoffs are 0

OUTLINE

STRATEGIC FORM GAMES

SOLVING A GAME: NASH EQUILIBRIUM

NASH EQUILIBRIUM

A strategy profile where no individual has a unilateral incentive to change her behavior

Before we talk about why this is our central solution concept, let's formalize it

NOTATION

Player i 's strategy

- ▶ s_i

Set of all possible strategies for Player i

- ▶ S_i

Strategy profile (one strategy for each player)

- ▶ $\mathbf{s} = (s_1, s_2, \dots, s_N)$

Strategy profile for all players except i

- ▶ $\mathbf{s}_{-i} = (s_1, s_2, \dots, s_{i-1}, s_{i+1}, \dots, s_N)$

Different notation for strategy profile

- ▶ $\mathbf{s} = (\mathbf{s}_{-i}, s_i)$

SELFISH STUDENTS

		Player 2	
		α	β
Player 1	α	1, 1	3, 0
	β	0, 3	2, 2

$$S_i = \{\alpha, \beta\}$$

4 strategy profiles: $(\alpha, \alpha), (\alpha, \beta), (\beta, \alpha), (\beta, \beta)$

CHICKEN

		Player 2	
		Straight	Swerve
Player 1	Straight	0, 0	3, 1
	Swerve	1, 3	2, 2

$$S_i = \{\text{Straight}, \text{Swerve}\}$$

4 strategy profiles: (Straight, Straight), (Straight, Swerve), (Swerve, Straight), (Swerve, Swerve)

CHOOSING A RESTAURANT

		Rebecca	
		P	V
Ethan	P	4, 3	1, 1
	V	0, 0	3, 4

$$S_E = ? \quad S_R = ?$$

Strategy profiles: ?

CHOOSING A NUMBER WITH 3 PLAYERS

$$S_i = [0, 10]$$

- ▶ Player i can choose any real number between 0 and 10

$$\mathbf{s} = (s_1 = 1, s_2 = 4, s_3 = 7) = (1, 4, 7)$$

- ▶ An example of a strategy profile

$$\mathbf{s}_{-2} = (1, 7)$$

- ▶ Same strategy profile, with player 2's strategy omitted

$$\mathbf{s} = (\mathbf{s}_{-2}, s_2) = ((1, 7), 4)$$

- ▶ Reconstructing the strategy profile

NOTATING PAYOFFS

Players' payoffs are defined over strategy profiles

- ▶ A strategy profile implies an outcome of the game

Player i 's payoff from the strategy profile \mathbf{s} is

$$u_i(\mathbf{s})$$

Player i 's payoff if she chooses s_i and others play as in \mathbf{s}_{-i}

$$u_i((s_i, \mathbf{s}_{-i}))$$

NASH EQUILIBRIUM

Consider a game with N players. A strategy profile $\mathbf{s}^* = (s_1^*, s_2^*, \dots, s_N^*)$ is a **Nash equilibrium** of the game if, for every player i

$$u_i(s_i^*, \mathbf{s}_{-i}^*) \geq u_i(s'_i, \mathbf{s}_{-i}^*)$$

for all $s'_i \in S_i$

BEST RESPONSES

A strategy, s_i , is a **best response** by Player i to a profile of strategies for all other players, \mathbf{s}_{-i} , if

$$u_i(s_i, \mathbf{s}_{-i}) \geq u_i(s'_i, \mathbf{s}_{-i})$$

for all $s'_i \in S_i$

BEST RESPONSE CORRESPONDENCE

Player i 's **best response correspondence**, BR_i , is a mapping from strategies for all players other than i into subsets of S_i satisfying the following condition:

- ▶ For each \mathbf{s}_{-i} , the mapping yields a set of strategies for Player i , $BR_i(\mathbf{s}_{-i})$, such that s_i is in $BR_i(\mathbf{s}_{-i})$ if and only if s_i is a best response to \mathbf{s}_{-i}

AN EQUIVALENT DEFINITION OF NE

Consider a game with N players. A strategy profile $\mathbf{s}^* = (s_1^*, s_2^*, \dots, s_N^*)$ is a **Nash equilibrium** of the game if s_i^* is a best response to \mathbf{s}_{-i}^* for each $i = 1, 2, \dots, N$

SELFISH VS. NICE

		Nice	
		α	β
Selfish	α	1, 2	3, 0
	β	0, 1	2, 3

SELFISH VS. NICE

		Nice	
		α	β
Selfish	α	1✓, 2	3, 0
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SELFISH VS. NICE

		Nice	
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Selfish	α	1✓, 2	3✓, 0
	β	0, 1	2, 3

SELFISH VS. NICE

		Nice	
		α	β
Selfish	α	$1^\checkmark, 2^\checkmark$	$3^\checkmark, 0$
	β	0, 1	2, 3

SELFISH VS. NICE

		Nice	
		α	β
Selfish	α	1✓, 2✓	3✓, 0
	β	0, 1	2, 3✓

SELFISH VS. NICE

		Nice	
		α	β
Selfish	α	1 [✓] , 2 [✓]	3 [✓] , 0
	β	0, 1	2, 3 [✓]

CHICKEN

		Player 2	
		Straight	Swerve
Player 1	Straight	0, 0	3, 1
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CHICKEN

		Player 2	
		Straight	Swerve
Player 1	Straight	0, 0	3, 1
	Swerve	1 [✓] , 3	2, 2

CHICKEN

		Player 2	
		Sterve	
Player 1	Straight	0, 0	3✓, 1
	Sterve	1✓, 3	2, 2

CHICKEN

		Player 2	
		Straight	Swerve
Player 1	Straight	0, 0	3 [✓] , 1 [✓]
	Swerve	1 [✓] , 3	2, 2

CHICKEN

		Player 2	
		Straight	Swerve
Player 1	Straight	0, 0	3 [✓] , 1 [✓]
	Swerve	1 [✓] , 3 [✓]	2, 2

CHICKEN

		Player 2	
		Straight	Swerve
Player 1	Straight	0, 0	3 [✓] , 1 [✓]
	Swerve	1 [✓] , 3 [✓]	2, 2

YOU SOLVE CHOOSING A RESTAURANT

		Rebecca	
		P	V
Ethan	P	4, 3	1, 1
	V	0, 0	3, 4

ANOTHER PRACTICE GAME

		Player 2	
		L	R
Player 1	U	10, 2	3, 4
	D	-1, 0	5, 7

WORKING IN A TEAM

$$u_1(s_1, s_2) = \pi(s_1, s_2) - s_1^2 = s_1 + s_2 + \frac{s_1 s_2}{2} - s_1^2$$

Find Player i 's best response by maximizing for each s_2

$$\frac{\partial u_1(s_1, s_2)}{\partial s_1} = 1 + \frac{s_2}{2} - 2s_1$$

WORKING IN A TEAM

$$u_1(s_1, s_2) = \pi(s_1, s_2) - s_1^2 = s_1 + s_2 + \frac{s_1 s_2}{2} - s_1^2$$

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$$\frac{\partial u_1(s_1, s_2)}{\partial s_1} = 1 + \frac{s_2}{2} - 2s_1$$

First-order condition sets this equal to 0 to get $BR_1(s_2)$

$$1 + \frac{s_2}{2} - 2BR_1(s_2) = 0$$

WORKING IN A TEAM

$$u_1(s_1, s_2) = \pi(s_1, s_2) - s_1^2 = s_1 + s_2 + \frac{s_1 s_2}{2} - s_1^2$$

Find Player i 's best response by maximizing for each s_2

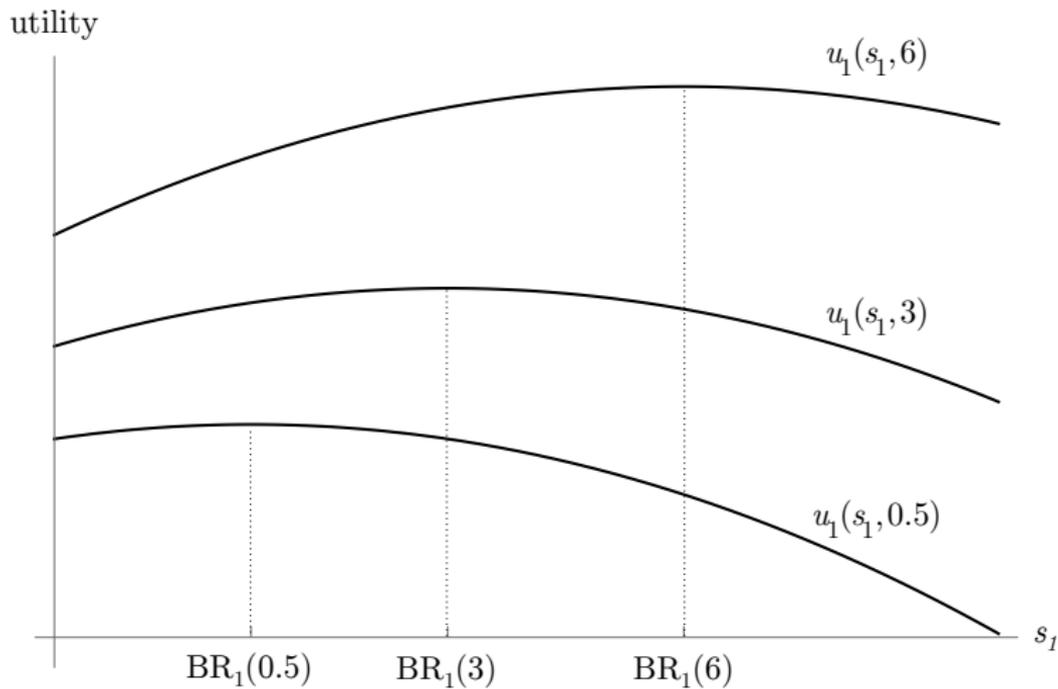
$$\frac{\partial u_1(s_1, s_2)}{\partial s_1} = 1 + \frac{s_2}{2} - 2s_1$$

First-order condition sets this equal to 0 to get $BR_1(s_2)$

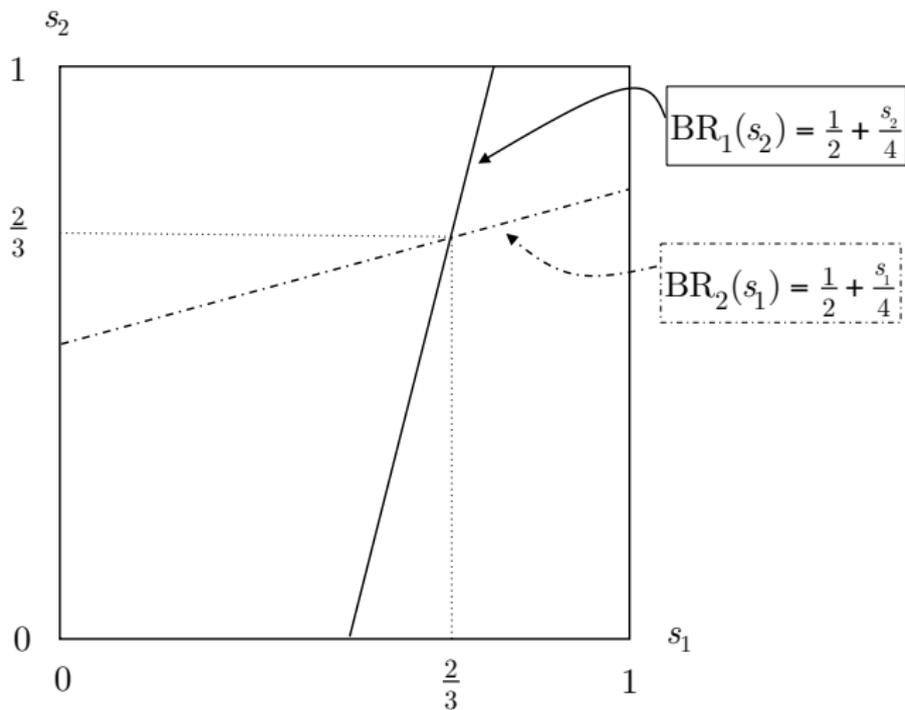
$$1 + \frac{s_2}{2} - 2BR_1(s_2) = 0$$

$$BR_1(s_2) = \frac{1}{2} + \frac{s_2}{4} \quad BR_2(s_1) = \frac{1}{2} + \frac{s_1}{4}$$

PLAYER 1'S BEST RESPONSE



NASH EQUILIBRIUM



SOLVING FOR NE

Since best responses are unique, a NE is a profile, (s_1^*, s_2^*) satisfying

$$s_1^* = \text{BR}_1(s_2^*) = \frac{1}{2} + \frac{s_2^*}{4} \quad s_2^* = \text{BR}_2(s_1^*) = \frac{1}{2} + \frac{s_1^*}{4}$$

Substituting

$$s_1^* = \frac{1}{2} + \frac{\frac{1}{2} + \frac{s_1^*}{4}}{4}$$

$$s_1^* = \frac{2}{3} \quad s_2^* = \frac{2}{3}$$

PRACTICE GAME WITH CONTINUOUS CHOICES

2 players

Each player, i , chooses a real number s_i

There is a benefit of value 1 to be divided between the players

At a strategy profile (s_i, s_{-i}) , Player i wins a share

$$\frac{s_i}{s_i + s_{-i}}$$

The cost of s_i is s_i

SOLVING

Write down Player 1's payoff from (s_1, s_2)

SOLVING

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$$u_1(s_1, s_2) = \frac{s_1}{s_1 + s_2} \times 1 - s_1$$

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Write down Player 1's payoff from (s_1, s_2)

$$u_1(s_1, s_2) = \frac{s_1}{s_1 + s_2} \times 1 - s_1$$

Calculate Player 1's best response correspondence

SOLVING

Write down Player 1's payoff from (s_1, s_2)

$$u_1(s_1, s_2) = \frac{s_1}{s_1 + s_2} \times 1 - s_1$$

Calculate Player 1's best response correspondence

$$\frac{\partial u_1(s_1, s_2)}{\partial s_1} = \frac{s_1 + s_2 - s_1}{(s_1 + s_2)^2} \times 1 - 1 = \frac{s_2}{(s_1 + s_2)^2} - 1$$

Set equal to zero to maximize

$$\frac{s_2}{(\text{BR}_1(s_2) + s_2)^2} - 1 = 0 \Rightarrow \text{BR}_1(s_2) = \sqrt{s_2} - s_2$$

SOLVING²

Player 2 is symmetric to Player 1, so write down both players' best response correspondences

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$$\text{BR}_1(s_2) = \sqrt{s_2} - s_2 \qquad \text{BR}_2(s_1) = \sqrt{s_1} - s_1$$

SOLVING²

Player 2 is symmetric to Player 1, so write down both players' best response correspondences

$$\text{BR}_1(s_2) = \sqrt{s_2} - s_2 \quad \text{BR}_2(s_1) = \sqrt{s_1} - s_1$$

At a NE each player is playing a best response to the other. Write down two equations that characterize equilibrium.

SOLVING²

Player 2 is symmetric to Player 1, so write down both players' best response correspondences

$$\text{BR}_1(s_2) = \sqrt{s_2} - s_2 \quad \text{BR}_2(s_1) = \sqrt{s_1} - s_1$$

At a NE each player is playing a best response to the other. Write down two equations that characterize equilibrium.

$$s_1^* = \sqrt{s_2^*} - s_2^* \quad s_2^* = \sqrt{s_1^*} - s_1^*$$

SOLVING³

$$s_1^* = \sqrt{s_2^*} - s_2^* \quad s_2^* = \sqrt{s_1^*} - s_1^*$$

Use substitution to find Player 1's equilibrium action

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$$s_1^* = \sqrt{s_2^*} - s_2^* \quad s_2^* = \sqrt{s_1^*} - s_1^*$$

Use substitution to find Player 1's equilibrium action

$$s_1^* = \sqrt{\sqrt{s_1^*} - s_1^*} - (\sqrt{s_1^*} - s_1^*) \Rightarrow s_1^* = \frac{1}{4}$$

Now substitute this in to find Player 2's equilibrium action

SOLVING³

$$s_1^* = \sqrt{s_2^*} - s_2^* \quad s_2^* = \sqrt{s_1^*} - s_1^*$$

Use substitution to find Player 1's equilibrium action

$$s_1^* = \sqrt{\sqrt{s_1^*} - s_1^*} - \left(\sqrt{s_1^*} - s_1^*\right) \Rightarrow s_1^* = \frac{1}{4}$$

Now substitute this in to find Player 2's equilibrium action

$$s_2^* = \sqrt{\frac{1}{4}} - \frac{1}{4} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

WHY NASH EQUILIBRIUM?

No regrets

Social learning

Self-enforcing agreements

Analyst humility

TAKE AWAYS

A Nash Equilibrium is a strategy profile where each player is best responding to what all other players are doing

You find a NE by calculating each player's best response correspondence and seeing where they intersect

NE is our main *solution concept* for strategic situations