Valence Competition and Platform Divergence*

Scott Ashworth† Ethan Bueno de Mesquita‡

First Version: February 12, 2007 This Version: July 24, 2007

Abstract

We study a game in which candidates for elected office first choose platforms and then invest in costly valences. The marginal return to valence accumulation depends on the degree of platform polarization—the closer together are the platforms, the more valence affects the election outcome. Consequently, even though candidates have no policy preferences, they diverge in equilibrium to soften valence competition. Moreover, exogenous increases in incentives for valence accumulation lead to both increased valence and increased polarization—the latter as a result of candidates seeking to avoid the costs of extra valence. As a result, the increase in valence is smaller in equilibrium than it would have been with exogenous platforms. Finally, the model highlights the overlooked substantive importance of common modeling assumptions. In particular, changing the source of uncertainty in our model from noise around the median voter's ideal point to a stochastic shock to one candidate's valence (as is common in the literature) leads to complete platform convergence for all parameter values.

*We are deeply indebted to Amanda Friedenberg for extensive comments and conversation. We have also benefited from comments by Alan Gerber, Don Green, Navin Kartik, Adam Meirowitz, Becky Morton, Tom Palfrey, Ken Scheve, and seminar audiences at Princeton and Yale.

†Department of Politics, Princeton University. Email: sashwort@princeton.edu

‡Harris School, The University of Chicago. Email: bdm@uchicago.edu

Candidates for elected office devote significant attention to at least two distinct types of campaign activities: establishing platform positions on issues, and building support on non-policy grounds (valence) by spending money on impressionistic advertising or building reputations for charisma.[1] Most existing models have treated these activities in isolation. (Stokes (1963) was an early critic of Downs (1957) on exactly this ground.) But this analytical approach runs the risk of missing important complementarities or substitutabilities between the two kinds of activity. For example, as we argue in this paper, the incentives to accumulate valence are likely to be sensitive to the degree of platform polarization. Thus a candidate might change her platform with an eye toward manipulating subsequent choices about valence accumulation.

The first strand of the literature to examine the interactions between platforms and valence focused on how a fixed valence advantage affects platform choices (Aragones and Palfrey, 2002; Ansolabehere and Snyder, 2000; Groseclose, 2001; Londregan and Romer, 1993; Schofield, Forthcoming). More recently, scholars have started to consider the joint determination of platforms and valences in equilibrium (Carrillo and Castanheira, 2006; Dickson and Scheve, 2006; Eyster and Kittsteiner, 2007; Herrera, Levine and Martinelli, 2005; Meirowitz, 2004; Morton and Myerson, 1992; Schofield, 2003; Zakharov, 2005). We contribute to this latter project. (In section 6.3, we relate our model to these other models of endogenous valence and platforms.)

We consider a game in which candidates first choose platforms and then choose valences. The voter's utility is additively separable between valence and policy, and the policy component is strictly concave in the distance between the implemented policy and the voter's ideal point. This implies that the marginal return to valence accumulation depends on the degree of platform polarization—the closer together are the platforms, the more the voter responds to valence. This endogenous weighting of valence and policy is the key innovation of the paper relative to the existing literature on the joint determination of valence and platforms.

Our first result is that parties diverge in equilibrium, even though they have no policy preferences. They diverge to soften valence competition. Intuitively, a candidate may be unwilling to move toward the median voter because the increase in the probability of winning is too small to compensate the candidate for the greater valence expenditures that

[1] The literature has discussed several different interpretations of valence. One prominent interpretation is that valence is built up by advertising or other campaign activities (Erikson and Palfrey, 2000). Polo (1998) studies a model in which politicians commit to levels of rent extraction prior to the election. In this case, valence is the inverse of corruption. Another interesting interpretation of valence is provided by Snyder and Ting's (2002) model of party discipline. In their model, discipline screens potential members of a party on the basis of ideology. Thus more disciplined parties have less dispersed distributions of candidate ideal points. Holding the mean ideology constant, this reduced dispersion acts as a valence term for risk-averse voters.

will follow.

To see why this matters, consider the extreme case of complete convergence. In this case, whichever candidate chooses the higher valence wins. Thus an arbitrarily small increase in valence can make the difference between losing and winning. This gives extreme incentives for valence accumulation. With divergence, on the other hand, a candidate's expected payoff is a continuous function of her valence, so a small increase in valence yields only a small increase in the chance of winning. More generally, greater platform polarization decreases the marginal benefit of valence accumulation.

The idea that candidates diverge to soften incentives for valence accumulation is reminiscent of the finding from industrial organization that firms might differentiate their products to soften price competition (Tirole, 1988, ch. 7). But there are some new subtleties involved in extending this idea to an electoral context. First, the winner-take-all nature of elections means that we must work with mixed-strategy equilibria in the valence subgames. While pure-strategy equilibria can fail to exist in Hotelling's pricing game for some specifications of transportation costs, pure-strategy existence can be guaranteed by appropriately choosing the functional form of transportation costs (d'Aspremont, Gabszewicz and Thisse, 1979; Anderson, 1988). Moreover, in the only paper we are aware of that treats the mixed strategy case, Osborne and Pitchik (1987) are unable to give a complete treatment of the subgame-perfect equilibria. Consequently, most of the IO literature focuses on the case of quadratic transportation costs, where pure-strategy existence is guaranteed.[2] Such a move is not open to us—there is no way to vary the functional forms to avoid mixed strategies in our model.

Second, extending the intuition about divergence highlights the substantive importance of modeling choices about the form of probabilistic voting in models of elections. Our result that polarization reduces the incentives to accumulate valence relies critically on the fact that uncertainty in our model is about the median voter's ideal point (what Duggan (2005) calls the "stochastic preference" model). Most of the political economy literature, on the other hand, uses a model in which uncertainty comes from an exogenous valence shock to one of the candidates (what Duggan calls the "stochastic partisanship" model). The literature's preference for the stochastic partisanship model is largely driven by the ease of working with it relative to the less well-behaved stochastic preference model. Our analysis certainly illustrates that the stochastic preference model can be hard to work with, but it also shows that the choice is not *simply* one of analytic convenience—substantive conclusions can be radically different between the two models. In particular, as we demonstrate formally in the section 6.1, changing our model from stochastic preferences to stochastic partisanship leads to complete platform convergence for all parameter values.

[2]Recent contributions in this vein are Anderson and Goeree (1997) and Meagher and Zauner (2004).

We also study how endogenous platforms affect the comparative statics of valence accumulation and platforms. Holding platforms fixed, an exogenous increase in the marginal benefit-to-cost ratio of valence accumulation would increase the amount of valence accumulated. However, platforms are not fixed in the model. In equilibrium, candidates, anticipating the greater investment in valence caused by an increase in the marginal benefit-to-cost ratio, may diverge more in order to soften valence competition. We show that, despite this countervailing effect associated with endogenous platforms, total equilibrium valence accumulation is increasing in the benefit-to-cost ratio of valence. However, endogenous platforms do have an important effect. In particular, the increase in valence accumulation is less than it would have been with exogenous platforms. Moreover, these two comparative statics suggest that, even though increased polarization decreases incentives for valence, we might observe a positive correlation between polarization and valence, since exogenous changes that increase incentives for valence also increase incentives for polarization.

The paper proceeds as follows. Section 1 introduces the formal model. Section 2 derives the voter's optimal voting rule and uses it to calculate the probability that a candidate wins for any profile of platforms and valences. Section 3 characterizes a particularly tractable class of mixed strategy equilibria in the valence accumulation subgame and demonstrates that, given platform locations, all such equilibria give rise to the same payoffs, which we calculate. Section 4 characterizes equilibrium platform locations. Section 5 discusses several comparative statics. Finally, Section 6 explores robustness and situates our model in the literature.

1 Model

A voter must choose one of two candidates. The voter cares about two attributes of a candidate: her policy platform $x \in \mathbb{R}$ and her valence $v \in \mathbb{R}_+$. The voter evaluates these two attributes according to the payoff function $u(x, v) = v - (x^* - x)^2$, where x^* is the voter's "ideal point". The voter's ideal point is unknown *ex-ante*—the common belief is that it is distributed uniformly on $[-\frac{\gamma}{2}, \frac{\gamma}{2}]$. The voter randomizes 50–50 if he is indifferent between the two candidates.

Denote the two candidates by L and R. Because the candidates do not have policy preferences, the labels are arbitrary. We always choose labels so that $x_L \leq x_R$.

A candidate, c, chooses both her platform and her valence to maximize

$$B \Pr(c \text{ wins}) - v_c.$$

Since payoffs are unique only up to an affine transformation, B should be interpreted as the ratio of benefit from winning to the marginal cost of valence.

The timing of the game is as follows.

1. The candidates simultaneously choose platforms x_c.
2. The candidates observe both platforms, and simultaneously choose valences v_c.
3. Nature determines the ideal point of the voter.
4. The voter observes the platforms, the realized valances, and his ideal point, and chooses a candidate.

This order of play is consistent, for example, with a model in which parties choose platforms at conventions and then candidates expend resources (i.e., accumulate valence) during campaigns that follow the conventions. Of course, other orders of play might also be interesting. In section 6.2 we discuss simultaneous choice of platforms and valences.

We look for subgame-perfect equilibria (SPE). Anticipating the results (where, in equilibrium, platforms are in pure strategies and valences are in mixed strategies), a strategy for a candidate is a pair (x_c, σ_c), where x_c is a platform and $\sigma_c(x_L, x_R)$ is a map

$$\sigma_c : \mathbb{R}^2 \to \Delta(\mathbb{R}_+)$$

giving candidate c's possibly random choice of valence in the valence subgame when x_L and x_R have been chosen as the platforms. We abuse notation and let (y, z) denote the subgame in which platforms are fixed at y for L and z for R. Then we write $\sigma_c^{y,z}$ for candidate c's measure on valences in the subgame (y, z).

We focus on SPE that are symmetric in the sense that $x_L = -x_R$ and $\sigma_L^{y,z} = \sigma_R^{-z,-y}$.

2 The Voting Subgame

The voter prefers L to R if

$$v_L - (x^* - x_L)^2 > v_R - (x^* - x_R)^2, \tag{1}$$

prefers R to L if the inequality is reversed and is indifferent if the two payoffs are equal.

Consider first the case of convergent platforms, $x_L = x_R$. If the valances are equal, then the voter is indifferent and randomizes 50–50. If the valances are not equal, then the voter votes for the advantaged candidate with probability 1.

Now consider profiles with $x_L \neq x_R$. Since the voter's payoff is supermodular in x^* and x, the optimal rule is a cutoff rule: vote L if and only if $x^* \leq \widehat{x}(x_L, x_R, v_L, v_R)$.

Straightforward algebra shows that

$$\widehat{x}(x_L, x_R, v_L, v_R) = \frac{1}{2}(x_R + x_L) + \frac{v_L - v_R}{2(x_R - x_L)}. \tag{2}$$

Notice that a valence advantage has more impact on the cutoff when the platforms are closer together. This is the key insight from this section. This result does not rely on the specific functional form assumptions we have made—a utility function that is additively separable in valence and the distance between the platform and ideal point has this implication as long as the utility is strictly concave in the distance between the platform and the ideal point.[3] We use the special quadratic form because it leads to a tractable solution that lets us explicitly construct the full equilibrium.

With this voting rule in hand, we can calculate the probabilities of winning given any configuration of platforms and valances. Let $\lambda(x_L, x_R, v_L, v_R)$ be L's probability of winning, and let $1-\lambda(x_L, x_R, v_L, v_R)$ be R's probability of winning. Also let F_γ be the cdf of a uniform $[-\frac{\gamma}{2}, \frac{\gamma}{2}]$ random variable:

$$F_\gamma(x) = \begin{cases} 1 & \text{if } x > \gamma/2 \\ \frac{x+\gamma/2}{\gamma} & \text{if } x \in [-\gamma/2, \gamma/2] \\ 0 & \text{if } x < -\gamma/2. \end{cases}$$

Then we have

$$\lambda(x_L, x_R, v_L, v_R) = \begin{cases} \frac{1}{2} & \text{if } x_L = x_R \text{ and } v_L = v_R \\ 1 & \text{if } x_L = x_R \text{ and } v_L > v_R \\ 0 & \text{if } x_L = x_R \text{ and } v_L < v_R \\ F_\gamma(\widehat{x}(x_L, x_R, v_L, v_R)) & \text{if } x_L \neq x_R. \end{cases}$$

The case of $x_L \neq x_R$ will be particularly important. We can substitute from equation 2 to get an explicit expression for the probability that L wins (assuming it is interior):

$$\lambda(x_L, x_R, v_L, v_R) = \frac{1}{2} + \frac{x_L + x_R}{2\gamma} + \frac{v_L - v_R}{2\gamma(x_R - x_L)},$$

[3] To see this, imagine that payoffs to the voter for candidate c are defined by $v_c - u(|x^* - x_c|)$ with $u(\cdot)$ increasing and strictly concave. The cutpoint $\hat{x}$ is implicitly defined by $v_L - u(|\hat{x} - x_L|) = v_R - u(|\hat{x} - x_R|)$. To see the intuition, suppose that $\hat{x} \in (x_L, x_R)$. Then we can rewrite the implicit definition of $\hat{x}$ as $u(x_R - \hat{x}) - u(\hat{x} - x_L) = v_R - v_L$. Implicitly differentiating shows that $\frac{\partial \hat{x}}{\partial (v_R - v_L)} = \frac{-1}{u'(x_R - \hat{x}) + u'(\hat{x} - x_L)} < 0$ and $\frac{\partial \hat{x}}{\partial x_L} = \frac{u'(\hat{x} - x_L)}{u'(x_R - \hat{x}) + u'(\hat{x} - x_L)} = 1 - \frac{\partial \hat{x}}{\partial x_R}$. Differentiating again yields $\frac{\partial^2 \hat{x}}{\partial (v_R - v_L)\partial x_R} = \frac{u''(x_R - \hat{x})(1 - \frac{\partial \hat{x}}{\partial x_R}) + u''(\hat{x} - x_L)\frac{\partial \hat{x}}{\partial x_R}}{(u'(x_R - \hat{x}) + u'(\hat{x} - x_L))^2} < 0$ and $\frac{\partial^2 \hat{x}}{\partial (v_R - v_L)\partial x_L} = -\frac{u''(x_R - \hat{x})\frac{\partial \hat{x}}{\partial x_L} + u''(\hat{x} - x_L)(1 - \frac{\partial \hat{x}}{\partial x_L})}{(u'(x_R - \hat{x}) + u'(\hat{x} - x_L))^2} > 0$ (where the inequalities follow from $u'' < 0$ and $0 < \frac{\partial \hat{x}}{\partial x_L} = 1 - \frac{\partial \hat{x}}{\partial x_R} < 1$). Thus, increasing x_R or decreasing x_L (i.e., increasing polarization) diminishes the impact of valence on $\hat{x}$.

if $x_L \neq x_R$ and $\hat{x}(x_L, x_R, v_L, v_R \in [-\gamma/2, \gamma/2]$.

3 Valence Subgame

The next step in rolling back is the valence accumulation subgame. In thinking about candidate L's incentives to accumulate valence, it is useful to consider a rearrangement of inequality 1. In particular, L wins the election if her valence satisfies the following inequality:

$$v_L > 2(x_R - x_L)x^* + x_L^2 - x_R^2 + v_R \equiv \mathcal{H}. \tag{3}$$

Thus, there is "hurdle" ($\mathcal{H}$) over which L's valence must lie in order for L to win. This hurdle is stochastic, because x^* is stochastic and v_R is stochastic if R plays a mixed strategy.

The variance of $\mathcal{H}$ affects the marginal benefit to L associated with increased valence. To see the intuition, consider the special case of a fixed v_R, so that the only source of variance in $\mathcal{H}$ is from x^*. In this case, $\mathcal{H}$ has a uniform distribution. Increasing the variance of $\mathcal{H}$ (i.e., increasing its support) decreases its density everywhere on the original support. As a result, when the variance of $\mathcal{H}$ increases, the marginal benefit of valence (in terms of increased probability of winning the election) is reduced. This point is illustrated in Figure 1, where the first panel shows benefit of increased valence with low variance and the second panel shows the (smaller) benefit of increased valence with higher variance.

Of course, v_R need not be fixed, since candidate R can play a mixed strategy. And for an arbitrary mixed strategy by R, the distribution of $\mathcal{H}$ will not be uniform. Nonetheless, the intuition developed above is useful because, as we show below, in the class of equilibria we study, candidate R plays a mixed strategy that results in $\mathcal{H}$ having a uniform distribution.

Given the intuition illustrated in Figure 1, equation 3 makes clear why platform divergence affects incentives for accumulating valence. The variance of $\mathcal{H}$ includes the variance of $2(x_R - x_L)x^*$, which is increasing in polarization. Thus, the marginal benefit of valence is decreasing in polarization.

This fact gives rise to following intuitions. For platforms that are fully converged, candidates have strong incentives for valence accumulation. We will see, in fact, that the valence subgame with full convergence gives each candidate a payoff of 0. For platforms that are highly diverged, players have no incentive to accumulate valence at all. At moderate levels of divergence, players have incentives to accumulate valence, but those incentives are weaker than in the case of complete convergence. The rest of this section formalizes exactly how these incentives play out.

We name accumulation subgames by the platform choices that lead to them.

Definition 1 *A* Nash equilibrium in the subgame (x_L, x_R) *is a pair of probability measures*

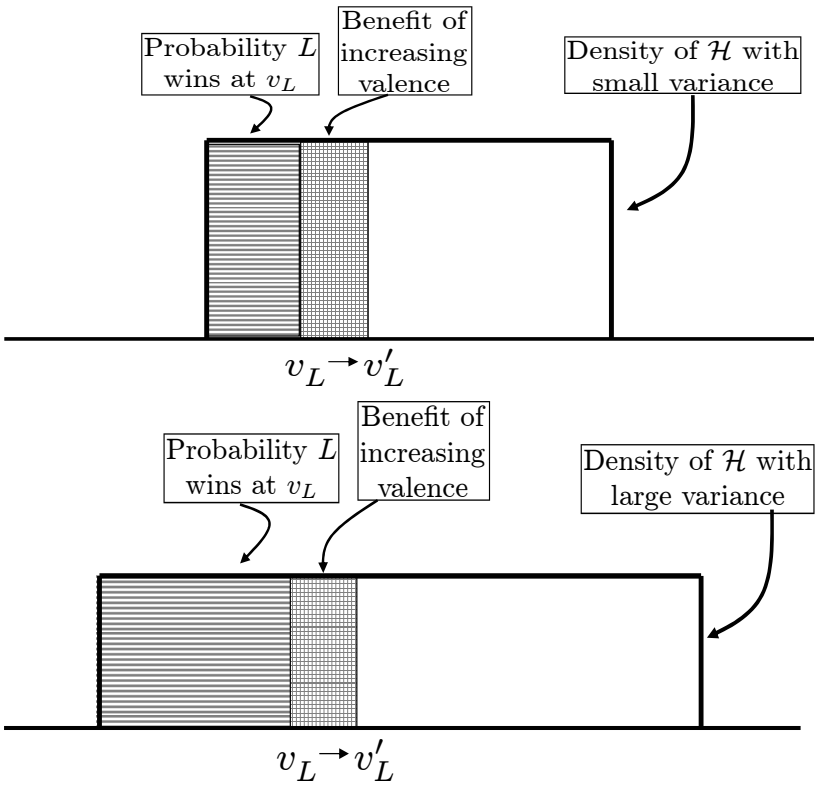

Figure 1: The marginal benefit of increased valence is decreasing in the variance of $\mathcal{H}$.

on $\mathbb{R}_+$*, denoted* $(\sigma_L^{x_L,x_R}, \sigma_R^{x_L,x_R})$*, such that:*

$$v \in \operatorname{supp} \sigma_L^{x_L,x_R} \Rightarrow v \in \arg\max \int \left(B\lambda(x_L, x_R, v, v') - v\right) \, d\sigma_R^{x_L,x_R}(v')$$

and

$$v \in \operatorname{supp} \sigma_R^{x_L,x_R} \Rightarrow v \in \arg\max \int \left(B\left(1 - \lambda(x_L, x_R, v', v)\right) - v\right) \, d\sigma_L^{x_L,x_R}(v').$$

The equilibrium characterization has three cases, covered in the next three subsections.

3.1 Complete Platform Convergence

Consider $x_L = x_R$. In this case, a candidate wins for sure if she has greater valence than the other candidate, and they both win with probability 1/2 if they choose the same valence. This subgame does not have a pure-strategy equilibrium. Any choice of valence strictly greater than B is strictly dominated. No profile with both valences strictly less than B can be an equilibrium because there must be at least one player who does not win for sure, and she could do better by increasing her valence to something slightly higher than her opponent's. No profile with unequal valences can be an equilibrium, because the high bidder can decrease her valence slightly and still win. Finally, both choosing B is not an equilibrium because the candidates then share the prize, and get a strictly negative payoff, whereas deviating to zero valence assures a payoff of at least zero.

Lemma 1 *Assume* $x_L = x_R$*. Then the unique equilibrium in the subgame has each candidate choosing valances according to a uniform distribution on* $[0, B]$*. Each candidate's payoff is 0.*

Proof The characterization is just Meirowitz's (2004) Proposition 3. To pin down the payoffs, note that, since the distributions are atomless the lowest valence level in the support (i.e., 0) loses for sure. Since the payoff from choosing zero valence and losing with certainty is 0, the indifference property of mixed strategy equilibria establishes the payoffs as 0. □

3.2 Highly Diverged Platforms

Whenever the platforms are not perfectly converged, the expected benefits to valence are

$$BF_\gamma\left(\frac{1}{2}(x_R + x_L) + \frac{v_L - v_R}{2(x_R - x_L)}\right)$$

for L and

$$B\left(1 - F_\gamma\left(\frac{1}{2}(x_R + x_L) + \frac{v_L - v_R}{2(x_R - x_L)}\right)\right)$$

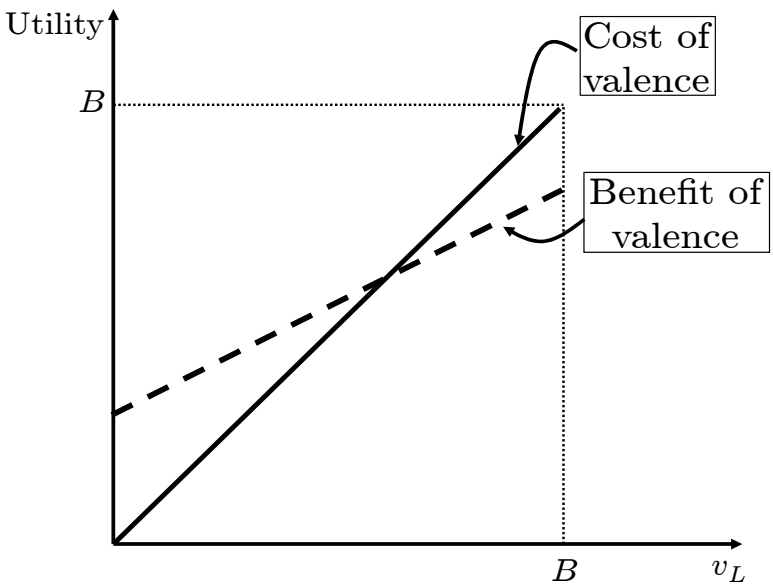

Figure 2: For highly diverged platforms, the payoff is decreasing in valence.

for R. These benefits are illustrated by the dashed line in Figure 2. These benefits can be thought of as the probability of winning given a level of valence (holding polarization and the other candidate's valence fixed) times the benefit of winning (B). The cost of valence is simply the level of valence chosen, illustrated by the solid 45-degree line in the figure.

As we discussed above, the size of the benefit to accumulating valence is a function of the level of platform divergence. The more diverged are the platforms, the smaller the return to valence (i.e., the shallower the slope of the dashed line). Thus, as platforms converge, the dashed line in Figure 2 rotates counter-clockwise.

The net expected benefit of any given level of valence is simply the dashed line minus the solid line. When platforms are sufficiently diverged (as in Figure 2) this net benefit is decreasing as valence increases. Consequently, the optimal choice is to accumulate no valence. This is true for any platform profile that is sufficiently diverged that the benefit line crosses the cost line from above.

As we will see in the next section, when platforms become sufficiently converged so that the benefit line crosses from below (what we refer to as "moderately diverged"), there will be incentives to accumulate valence.

The following result formalizes the intuition that highly diverged platforms lead to zero valence and establishes the precise level of divergence where the incentives switch ($x_R - x_L = \frac{B}{2\gamma}$).

Lemma 2 *If $2\gamma(x_R - x_L) \geq B$, then $v_L = v_R = 0$ is an equilibrium in the subgame. If the*

inequality is strict, then the equilibrium is unique.

(All proofs not in the text are in the appendix.)

3.3 Moderately Diverged Platforms

If platforms are moderately diverged ($(x_R - x_L) < \frac{B}{2\gamma}$), candidates have incentives to accumulate valence. However, there is not an equilibrium of the valence accumulation subgame with moderately diverged platforms in either pure strategies or non-atomic mixed strategies (as in the complete convergence case).

Figure 3 illustrates the benefits and costs to valence when platforms are only moderately diverged. The intuition of the figure is as follows. Since platforms are only moderately diverged, the line representing the benefit of valence is rotated sufficiently far in the counter-clockwise direction that the net benefit of valence is increasing in the level of valence (whereas it was decreasing in Figure 2). There are also flat regions on both the left and right tails of the benefit line. These flat regions result from the fact that the median voter's ideal point has bounded support. Consequently, for sufficiently low valence accumulation by the left-wing candidate, there is no ideal point that the voter could have that would lead him to vote for the left candidate. This is represented by the flat region on the benefit line's left tail. Similarly, for sufficiently large levels of valence accumulation, there is no ideal point that the voter could have that would lead him to vote against the left candidate. This is represented by the flat region on the benefit line's right tail. Between these two flat regions, the left candidate wins with increasing probability as she increases her valence accumulation.

Given this, it is fairly straightforward to see why there are no pure strategy equilibria. When platforms are moderately diverged, as in Figure 3, the *net* benefit of valence accumulation is strictly increasing for any valence that gives any non-degenerate probability of winning. Moreover, it is maximal at the point where valence is just great enough that the candidate wins for certain, holding the other candidate's valence fixed (i.e., at the point where the right tail flat region begins). This point is labeled $\hat{v}_L$. Hence, each candidate wishes to accumulate just enough valence that she wins for certain, holding fixed the other candidate's valence. Clearly, it is not possible for both candidates to do this. Thus, there cannot be a pure strategy equilibrium. This intuition is formalized in the following result.

Lemma 3 *If $2\gamma(x_R - x_L) < B$ and $x_R \neq x_L$, then*

1. *An equilibrium of the valence subgame exists;*

2. *There is no pure strategy equilibrium in the valence subgame;*

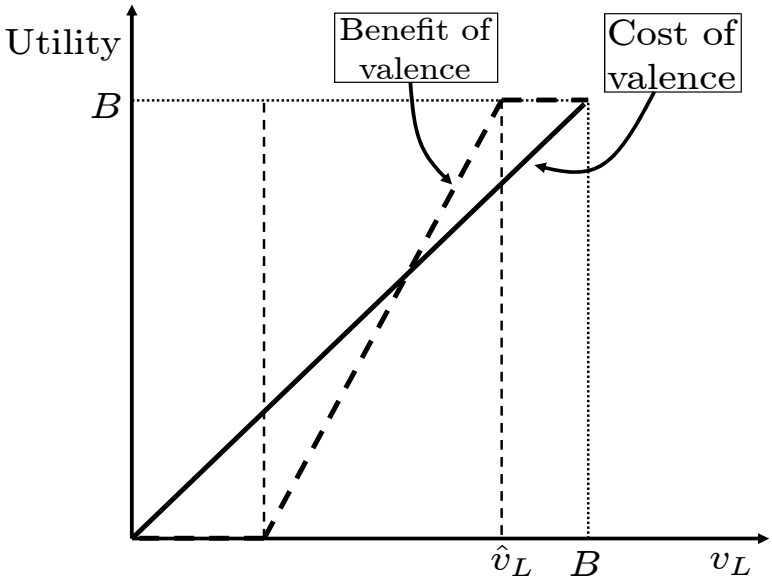

Figure 3: For moderately diverged platforms, the payoff from valence is maximized at $\hat{v}_L$, the valence where the candidate just wins for certain (holding the other candidate's valence fixed).

3. There is no non-atomic mixed strategy equilibrium.

We can begin to build some intuition for what a mixed-strategy equilibrium looks like by considering the first-order condition that must be satisfied by any $v > 0$ in the support of candidate L's mixed strategy. This first order condition is

$$\frac{\partial}{\partial v_L} \int \left(BF_\gamma \left(\frac{1}{2}(x_R + x_L) + \frac{v_L - v_R}{2(x_R - x_L)} \right) - v_L \right) d\sigma_R^{x_L, x_R}(v_R) = 0.$$

It will be useful to introduce a bit more notation. In particular, let $I(v_c, x_L, x_R)$ be the interval such that, for a given v_c, x_L, and x_R, a choice of v_{-c} in $I(v_c, x_l, x_R)$ implies that both candidates win with positive probability. For example, the interval between the vertical dashed lines in Figure 3 is $I(v_R, x_L, x_R)$. That is, for the v_R fixed in the figure, any choice of v_L between the vertical dashed lines leads to a positive probability of either candidate winning.

If v_L is a best response and R's mixture assigns probability zero to neighborhoods of the endpoints of I, then we can differentiate under the integral and use the definition of F_γ to rewrite the first order condition as

$$\Pr\left(v_R \in I(v_L, x_L, x_R)\right) \frac{B}{2\gamma(x_R - x_L)} = 1.$$

Thus, if v_L is in the support of L's mixed strategy, then R's mixed strategy must assign probability of exactly $\frac{2\gamma(x_R - x_L)}{B}$ to the interval $I(v_L, x_L, x_R)$.

Let $\alpha \equiv 2\gamma(x_R - x_L)$ be the length of this interval. It represents the smallest increase in valence that can change a candidate from being a sure loser into a sure winner. Notice, again, the more polarized are the parties (the greater is $x_R - x_L$), the more valence it takes for a candidate to swing the election in his favor.

We look for mixed-strategy equilibria in a particularly tractable class of profiles.

Definition 2 *A profile is* non-overlapping in the valence subgame *if*

1. *it has finite support;*
2. *puts positive mass on zero;*
3. *for each interval $I(v_L, x_L, x_R)$ for $v_L > 0$ contains exactly one v in the support of R's mixed strategy and;*
4. *for each interval $I(v_R, x_L, x_R)$ for $v_R > 0$ contains exactly one v in the support of L's mixed strategy.*

Definition 3 *A profile is* non-overlapping in the overall game *if, for every valence accumulation subgame that does not have convergent platforms, play is non-overlapping in the valence subgame.*

A non-overlapping profile satisfies all of the first-order conditions for non-zero atoms in L's mixture exactly if each non-zero atom in R's mixture has probability $\frac{\alpha}{B}$. The simplest way to achieve this is to let each player assign probability $\frac{\alpha}{B}$ to a finite series of vs, each α apart from one another. There is a complication to this plan—since the number of vs must be an integer, and each receives probability $\frac{\alpha}{B}$, this procedure yields probabilities that add up to 1 only if B/α is an integer. To solve this problem, we fill up as much of the probability as we can with interior atoms, and put the rest somewhere where the first-order condition does not have to hold with equality—at $v = 0$.[4] The definition imposes the additional requirement that there be positive mass on zero even if it were to happen that there was no left-over probability (i.e., if B/α was an integer). Without this requirement, the function mapping platform choices into payoffs in the subgame could fail to be upper semi-continuous.

The following notation is useful for dealing with this extra probability. Let $\lceil x \rceil$ be the least integer greater than or equal to x, and let $T(x) = \lceil x \rceil - 1$.[5] Thus $(1 - \alpha/B \cdot T(B/\alpha))$

[4] As we will see, there is not a unique way to carry out the construction.

[5] We use $T(x)$ rather than $\lfloor x \rfloor$ to ensure that the payoff function we derive later is upper semi-continuous.

is the amount of probability left over after we have created as many atoms of probability α/B as possible.

The next result puts some structure on any possible non-overlapping mixed-strategy equilibrium.

Lemma 4 *Let (σ_L, σ_R) be a non-overlapping mixed strategy equilibrium of the subgame (x_L, x_R). Then*

1. *Each non-zero atom of a candidate's mixture has probability $\frac{\alpha}{B}$;*
2. *Each mixture has $n = T(B/\alpha) + 1$ points in its support; and*
3. *Each mixture gives acquiring zero valence probability p such that*

$$0 < p = 1 - \frac{\alpha}{B} T\left(\frac{B}{\alpha}\right) \leq \frac{\alpha}{B}.$$

Label the elements of the support of a candidate's mixed strategy $\mathcal{V}_c = \{v_c^1, v_c^2, \ldots, v_c^n\}$, where

$$0 = v_c^1 < v_c^2 < \cdots < v_c^n.$$

Further, label the probability measure on $\mathcal{V}$ that constitutes a candidate's mixed strategy $(p_c^1, p_c^2, \ldots, p_c^n) = \left(1 - \frac{\alpha}{B} T\left(\frac{B}{\alpha}\right), \frac{\alpha}{B}, \frac{\alpha}{B}, \ldots, \frac{\alpha}{B}\right)$, where the equality follows from Lemma 4.

In a mixed strategy equilibrium, each candidate must be indifferent over all of these possible valences. This implies a precise relationship between the pace at which valences increase and the various probabilities that L wins.

Lemma 5 *Let β^k be the probability that L wins when the valences are v_L^k and v_R^k. In a non-overlapping mixed strategy equilibrium, the following conditions must hold:*

1. $v_L^2 = Bp_L^1\beta^1 + \alpha(1 - \beta^1)$
2. $v_R^2 = Bp_R^1(1 - \beta^1) + \alpha\beta^1$
3. $k > 1$ *implies*

$$v_L^{k+1} - v_L^k = \alpha + \alpha(\beta^{k+1} - \beta^k)$$

and

$$v_R^{k+1} - v_R^k = \alpha - \alpha(\beta^{k+1} - \beta^k).$$

In any non-overlapping equilibrium, each candidate plays zero with strictly positive probability. By the indifference property of mixed strategy equilibria, the equilibrium payoff must be the payoff to playing zero, and non-overlappingness implies that this payoff is just the probability that the other player plays zero times the probability of victory given no valence is accumulated at all. And that quantity is pinned down independently of the details of the rest of the equilibrium. This argument is used to prove the second part of the following result.

Proposition 1 *1. For any $x_L \neq x_R$ with $\alpha = 2\gamma(x_R - x_L) < 1$, there exists a non-overlapping mixed strategy equilibrium in the subgame (x_L, x_R).*

2. In any non-overlapping mixed strategy equilibrium of the subgame (x_L, x_R), payoffs are

$$\pi_L(x_L, x_R) = B\left(\frac{1}{2} + \frac{x_R + x_L}{2\gamma}\right)\left(1 - \frac{2\gamma(x_R - x_L)}{B}\, T\left(\frac{B}{2\gamma(x_R - x_L)}\right)\right)$$

to candidate L and

$$\pi_R(x_L, x_R) = B\left(\frac{1}{2} - \frac{x_R + x_L}{2\gamma}\right)\left(1 - \frac{2\gamma(x_R - x_L)}{B}\, T\left(\frac{B}{2\gamma(x_R - x_L)}\right)\right).$$

to candidate R.

3. These payoffs (π_L and π_R) are strictly positive.

The conditions of Lemmata 4 and 5 are *not* enough to pin down a unique non-overlapping mixed strategy equilibrium. And indeed, there can be multiple such equilibria.

To get a sense of how we construct an equilibrium and why it is not unique, consider an example of the valence accumulation subgame in which $x_L + x_R = 0$, $\alpha = 2/5$, and $B = 1$. This example will show the steps we use in constructing an equilibrium. It will also demonstrate exactly where multiplicity can occur, why the payoffs are the same for any non-overlapping equilibrium, and what choice we make for constructing our particular equilibrium.

Lemma 4 implies that if $\alpha = 2/5$ and $B = 1$, then $p_c^1 = 1/5$, $v_c^1 = 0$, and $p_c^2 = p_c^3 = 2/5$ for both candidates. This further implies that $\beta^1 = 1/2$. Points 1 and 2 of Lemma 5 and the definition of β^2 now yield the following system of three equations with three unknowns:

$$\begin{aligned} v_L^2 &= Bp_L^1\left(\frac{1}{2} - \frac{x_R + x_L}{2\gamma}\right) + Bp_L^2\beta^2 \\ &= \frac{1}{10} + \frac{2\beta^2}{5}, \end{aligned}$$

$$
\begin{aligned}
v_R^2 &= Bp_R^1\left(\frac{1}{2}+\frac{x_R+x_L}{2\gamma}\right)+Bp_R^2(1-\beta^2)\\
&= \frac{1}{10}+\frac{2(1-\beta^2)}{5},
\end{aligned}
$$

and

$$
\begin{aligned}
\beta^2 &= \frac{1}{2}+\frac{x_R+x_L}{2\gamma}+\frac{v_L^2-v_R^2}{\alpha}\\
&= \frac{1}{2}+\frac{5(v_L^2-v_R^2)}{2}.
\end{aligned}
$$

Solving this system gives $v_L^2 = v_R^2 = \frac{3}{10}$ and $\beta^2 = \frac{1}{2}$.

So far, we know that in any non-overlapping mixed strategy equilibrium, both candidates choose $p_c^1 = 1/5$, $p_c^2 = p_c^3 = 2/5$, $v_c^1 = 0$, and $v_c^2 = 3/10$. However, this does not pin down the choice of v_c^3.

From point 3 of Lemma 5, we know that

$$v_L^3 - v_L^2 = \alpha + \alpha(\beta^3 - \beta^2),$$

and

$$v_R^3 - v_R^2 = \alpha - \alpha(\beta^3 - \beta^2),$$

Substituting and rearranging yields:

$$v_L^3 = \frac{1}{2}+\frac{2}{5}\beta^3,$$

and

$$v_R^3 = \frac{1}{2}+\frac{2}{5}(1-\beta^3).$$

Adding the two conditions together we have

$$v_L^3 + v_R^3 = \frac{7}{5}.$$

Since we are looking for non-overlapping equilibria, we also need

$$v_L^3 \geq \max I(v_R^2, x_L, x_R) = v_R^2 + \frac{\alpha}{2} + (x_R^2 - x_L^2) = \frac{3}{10}+\frac{2}{5}\cdot\frac{1}{2} = \frac{1}{2}$$

and

$$v_R^3 \geq \max I(v_L^2, x_L, x_R) = \frac{1}{2}.$$

Now it is clear that for any $v_L^3 \in [1/2, 9/10]$, the pair $(v_L^3, 7/5 - v_L^3)$ completes the specification of a mixed strategy equilibrium. In the proof of Proposition 1, we construct the MSE in which $\beta^3 = \beta^2$, which, in this case, implies that $v_L^3 = v_R^3 = 7/10$. The important point, however, is that all equilibria in this family produce the same expected payoffs, 1/10 for each candidate.

The above example highlights the fact that, in addition to being able to characterize equilibrium valence accumulation, we can also calculate the total valence accumulated by the two candidates. Such a result will be interesting later because it will allow us to think about how exogenous changes to the electoral environment affect the interaction between platform divergence and valence.

Proposition 2 *Fix B, γ, and platforms x_L and $x_R = -x_L$ such that $B > 2\gamma(x_R - x_L) > 0$. Then the expected sum of valences in any non-overlapping, mixed strategy, Nash equilibrium of the subgame is*

$$2\gamma(x_R - x_L)T\left(\frac{B}{2\gamma(x_R - x_L)}\right).$$

It is worth noting that our characterization of equilibrium in the valence accumulation subgame is similar to the equilibrium for the linear contest characterized by Che and Gale (2000). The key difference is as follows. Since platform choices are potentially asymmetric, the probability of victory in the valence subgame of our model can be asymmetric. Hence, in order to characterize equilibrium in all valence subgames, we have to consider asymmetries that do not appear in Che and Gale (2000), which does not have a previous stage and, hence, focuses on contests where the probability of winning function is symmetric.

4 Platforms

The final step in finding an equilibrium is to characterize the candidates' platform choices, given equilibrium behavior in the subgames.

4.1 Divergence

Our first result is that *no* equilibrium has convergent platforms, even though the candidates have no policy preferences. The reason is apparent from looking at the voter's optimal election rule. When the platforms are very close together, even small valence differences loom large in the voter's decision. In the case of complete convergence, any difference in valances completely determines the election. This leads the competition over valence to be so intense that all of the gains are dissipated, and the candidates end up with payoffs of

zero. Anticipating this, they will not converge. (Meirowitz (2004) proves a similar result with a three-point policy space.)

Proposition 3 *There is no subgame-perfect equilibrium with convergent platforms ($x_L = x_R$).*

Proof Recall that the unique equilibrium has each candidate choosing valances according to a uniform distribution on $[0, B]$. Since the distributions are atomless, the lowest bid loses for sure. This pins down the payoffs at 0. But the payoff functions given in Proposition 1 are strictly positive, so a sufficiently small move away from convergence is a profitable deviation. □

4.2 Characterization

We have established a result about what *cannot* happen in equilibrium—the parties do not converge, even though they have no policy preferences. The key to understanding which platforms are chosen in equilibrium is to get a good understanding of the payoff function

$$\pi_L(x_L, x_R) = B\left(\frac{1}{2} + \frac{x_R + x_L}{2\gamma}\right)\left(1 - \frac{2\gamma(x_R - x_L)}{B}\, T\left(\frac{B}{2\gamma(x_R - x_L)}\right)\right).$$

(By symmetry, it suffices to look only at the left-most candidate.) The first factor, $B(1/2 + (x_R + x_L)/2\gamma)$, is increasing in x_L. Furthermore, the second factor is "usually" increasing in x_L as well. To see this, notice that $\frac{x_R+x_L}{2\gamma}(1 - \frac{2\gamma(x_R-x_L)}{B})$ is increasing in x_L and $T(B/2\gamma(x_R - x_L))$ is constant in x_L at every point where $B/(2\gamma(x_R - x_L))$ is not an integer. However, as x_L increases, at each point where $B/(2\gamma(x_R - x_L))$ hits an integer, $T\left(\frac{B}{2\gamma(x_R-x_L)}\right)$ jumps down discontinuously. Thus, the payoff function is almost everywhere increasing, but has a series of discontinuous jumps down, as illustrated in Figure 4.

As the figure makes clear, it is precisely at the platform locations that coincide with the discontinuity points (i.e., that make $B/(2\gamma(x_R - x_L))$ an integer) that the candidates' payoffs reach local maxima. This observation gives us the following fact about equilibrium platforms.

Lemma 6 *If x_L is a best response to x_R, then $\frac{B}{2\gamma(x_R-x_L)}$ is an integer.*

The fact that the possible optimal platform choices all coincide with the discontinuity points suggests a way to identify optimal platforms without having to work directly with the poorly-behaved payoff function. In particular, we can define a function $f(x_L, x_R)$ that bounds the payoff function from above and is equal to the payoff function at each local

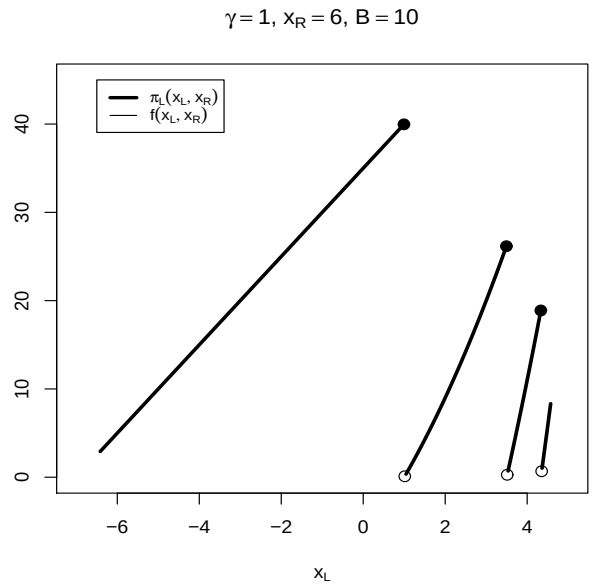

Figure 4: As the left candidate's platform increases, his equilibrium payoff is increasing except at a countable number of discontinuities, where it jumps down.

maximum. This function turns out to be the following well-behaved function:

$$f(x_L, x_R) = \left(\frac{1}{2} + \frac{x_R + x_L}{2\gamma}\right)(2\gamma(x_R - x_L)).$$

Note that f is strictly concave in x_L.

Lemma 7 *Assume that candidate L wins with strictly positive probability when both valences are zero ($x_L \geq -\gamma - x_R$). Then*

1. $\pi_L(x_L, x_R) \leq f(x_L, x_R)$
2. $\pi_L(x_L, x_R) = f(x_L, x_R)$ *if and only if* $\frac{B}{2\gamma(x_R - x_L)}$ *is an integer.*

As illustrated in Figure 5, the previous two lemmata show that all potential best responses are points on a strictly concave function.[6] This makes the task of checking for equilibrium in platforms relatively simple—we only need to check for deviations to adjacent candidates for a best response.

Now we have all of the ingredients we need to construct an equilibrium in platforms. By Lemma 6, the distance between the platforms, $x_R - x_L$, must make the ratio $\frac{B}{2\gamma(x_R - x_L)}$

[6]Recall that, in a non-overlapping equilibrium, a candidate's payoff in a subgame is her payoff from choosing zero valence in that subgame. Thus it cannot be a best response to choose a platform that makes Lemma 7 not apply.

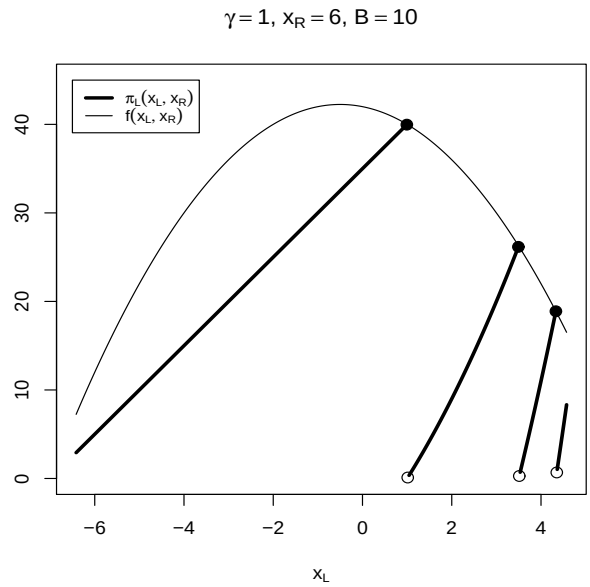

Figure 5: The payoff function (π_L) is bounded above by the strictly concave function f. Moreover, because the discontinuities occur precisely when $\pi_L = f$, all platforms for candidate L that are potential best responses lie on f.

an integer, say n. Thus the platforms must be $\frac{B}{2\gamma n}$ apart. And since we are looking for a symmetric equilibrium, the platforms must be

$$x_L = -\frac{B}{4\gamma n} \quad \text{and} \quad x_R = \frac{B}{4\gamma n}.$$

Proposition 4 *For any values of the parameters, there exists a non-overlapping equilibrium of the overall game. If $B \geq 2\gamma^2$, then there is a non-overlapping equilibrium of the overall game with moderately diverged platforms, and if $B > 4\gamma^2$, then all non-overlapping equilibria of the overall game have moderately diverged platforms.*

Proof The key step in the proof is the following construction.

Lemma 8 *Fix a positive integer n. The profile in which:*

1. *The platforms are*
$$x_L^* = -\frac{B}{4n\gamma} \quad \textit{and} \quad x_R^* = \frac{B}{4n\gamma}.$$

2. *Candidate play the valence subgame as follows:*

   - *If $x_L = x_R$, each candidate chooses valances according to a uniform distribution on $[0, B]$.*

- *If $x_L \neq x_R$ and $2\gamma(x_R - x_L) > B$, then the candidates choose valences $v_L = v_R = 0$.*
- *Otherwise, the candidates choose valence according to the profile constructed in the proof of Proposition 1.*

3. A voter votes for candidate L if an only if $x^* \leq \hat{x}$, where $\hat{x}$ is defined in equation (2).

is a subgame perfect equilibrium if and only if

$$\frac{1}{2}\frac{B}{\gamma^2} - 1 \leq n \leq \frac{1}{2}\frac{B}{\gamma^2} + 1.$$

Furthermore, any symmetric non-overlapping equilibrium must have platforms as in 1.) for some integer $n \geq 1$.

The proof of Lemma 8 is in the appendix.

To prove the existence assertion, notice that the interval $[B/(2\gamma^2) - 1, B/(2\gamma^2) + 1]$ has length 2, and is thus guaranteed to contain an integer n, and that

$$\inf_{B \geq 0, \gamma \geq 0} \frac{B}{2\gamma^2} + 1 = 1,$$

so we can take $n \geq 1$. Since the profile constructed in the Lemma is a non-overlapping equilibrium, such an equilibrium exists for any parameter values.

Finally, note that a profile of the form 1–3 has moderately diverged platforms if and only if $n \geq 2$. Thus there exists a moderately diverged non-overlapping equilibrium if

$$\frac{B}{2\gamma^2} + 1 \geq 2,$$

or $B \geq 2\gamma^2$. Similarly, all non-overlapping equilibria are moderately diverged if

$$\frac{B}{2\gamma^2} - 1 > 1,$$

or $B > 4\gamma^2$. □

5 Comparative Statics

In this section we explore the joint comparatives statics of valence accumulation and platform choice. In doing so, we highlight the way that allowing both valence and platforms to be endogenous changes the empirical predictions based on models with exogenous platforms.

In models with endogenous valence, but exogenous platforms, a condition like equation 3 still characterizes the voter's strategy. As is clear from that equation, an exogenous increase in platform divergence decreases incentives for valence accumulation. Thus, Ashworth and Bueno de Mesquita (2006) and Ashworth and Bueno de Mesquita (forthcoming) find that exogenous increases in platform divergence decrease incentives for local public goods provision and for investing in an informative party label (both forms of endogenous valence advantages).

Here it is also true that platform divergence mitigates incentives for valence accumulation. However, in contrast to models with exogenous platforms, the model in this paper suggests that we may expect to see a positive, rather than a negative, correlation between platform divergence and valence. The reason is that, as shown below, an exogenous increase in B increases incentives for divergence and for valence. And the increase in divergence is not enough to completely offset the incentives for increased valence. Thus, as B increases, both valence and divergence tend to increase in equilibrium.

The key point here is not that our model definitively demonstrates that the correlation between platforms and valence is expected to be positive. Rather, there are competing effects, and which one dominates remains uncertain.

5.1 Comparative Statics of Valence

Proposition 2 reports the total expected valence accumulation in any symmetric, subgame perfect equilibrium with a non-overlapping equilibrium in the valence accumulation subgame. From this calculation, it is clear that the expected amount of valence accumulated for fixed platforms is nondecreasing in B, the ratio of benefits to winning office over the marginal cost of accumulating valence.

Of course, in equilibrium, platforms are not fixed—they change endogenously as B changes. In particular, as B increases, the candidates may diverge further in order to diminish the anticipated increase in valence spending caused by an increase in B. Nonetheless, the fact that valence accumulation is non-decreasing in B remains true with endogenous platforms. However, the fact that the candidates can endogenously diverge implies that an increase in B leads to a smaller increase in valence than would have occurred if platforms were fixed. The differing effects of an increase in B on total expected valence accumulation with fixed and endogenous platforms are illustrated in Figure 6.

To see a somewhat more technical intuition, consider parameters B and γ and platforms x_L and x_R that are part of a symmetric, non-overlapping subgame perfect equilibrium for

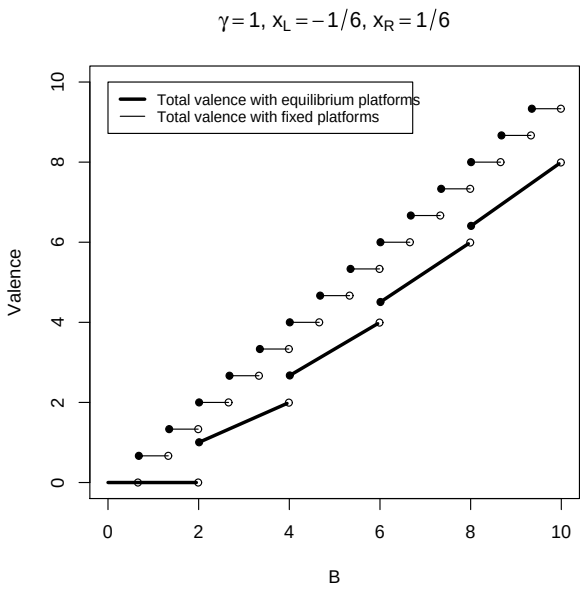

Figure 6: As the ratio of benefits to winning office over marginal costs of valence accumulation increases, total expected valence accumulation increases. This increase is larger when platforms are fixed than when platforms are endogenous.

those parameters. By Proposition 2, the expected valence is

$$2\gamma(x_R - x_L)T\left(\frac{B}{2\gamma(x_R - x_L)}\right).$$

If B increases slightly, equilibrium platforms adjust so that $B/2\gamma(x_R - x_L)$ stays constant. In particular, point 3 of Lemma 8 shows that, in equilibrium, $x_R - x_L = \frac{B}{2\gamma n}$. Substituting for these equilibrium platforms shows that total equilibrium valence accumulation is

$$\frac{n-1}{n}B.$$

Thus, an increase from B to $B + \epsilon$, for ϵ small, leads to an increase in valence of

$$\epsilon\frac{n-1}{n}.$$

What would happen if platforms could not adjust? Recall that equilibrium platforms make the ratio $\frac{B}{2\gamma(x_R - x_L)}$ an integer. Thus, without endogenous platforms, increasing B slightly makes $T\left(\frac{B}{2\gamma(x_R - x_L)}\right)$ step up by 1, and the increase in valence accumulation is

$$2\gamma(x_R - x_L).$$

Clearly, for small ϵ, $\epsilon\frac{n-1}{n} < 2\gamma(x_R - x_L)$, so valence increases less with endogenous platforms than it would with exogenous platforms.

5.2 Comparative Statics of Platforms

The discussion above suggests that as the benefit to cost ratio of valence increases, platforms diverge in order mitigate the increase in valence accumulation. However, just as there are discontinuities in the comparative static on valence accumulation, there are also discontinuities in the comparative static on platforms.

In particular, platform divergence in any given equilibrium is given by $\frac{B}{2n\gamma}$. As B increases, as long as n stays fixed, divergence increases continuously. However, for the same reason that we had the discontinuities above, at some point B increases so much that n increases discontinuously. Thus, at these points, we have a continuous increase in the numerator and a discontinuous increase in the denominator, leading to a discontinuous decrease in divergence. Of course, after this jump toward convergence, as B continues to increases, divergence returns to increasing until the next discontinuity.

The question remains whether the discontinuous jumps toward convergence are large enough that, as B gets very large, platforms tend toward convergence or divergence. It turns out that they tend toward divergence.

To see this, first notice that divergence is decreasing in n. Moreover, Lemma 8 shows that in any equilibrium $n \leq \frac{1}{2}\frac{B}{\gamma^2} + 1$. Thus, platform convergence is maximized by selecting the equilibrium that keeps n as close to its upper bound as possible. We study this equilibrium and show that even in this worst case, as B gets large, platforms tend toward greater divergence.

In this equilibrium, the variable n jumps from $k-1$ to k at $B = 2\gamma^2(k-1)$. We want to compare the level of divergence at the discontinuity where n moved from $k-2$ to $k-1$ to the level of divergence at the discontinuity where n moved from $k-1$ to k. If the latter is larger, that means that as B increases, the discontinuous jump in convergence does not make up for the previous increase in divergence. At the point where n moves from $k-2$ to $k-1$, we can substitute the value of B at the discontinuity into the formula for divergence to find that divergence is given by:

$$\gamma\frac{k-2}{k-1}.$$

At the point where n moves from $k-1$ to k, divergence is given by:

$$\gamma\frac{k-1}{k}.$$

The latter is clearly larger, so the discontinuous jumps toward convergence are not making

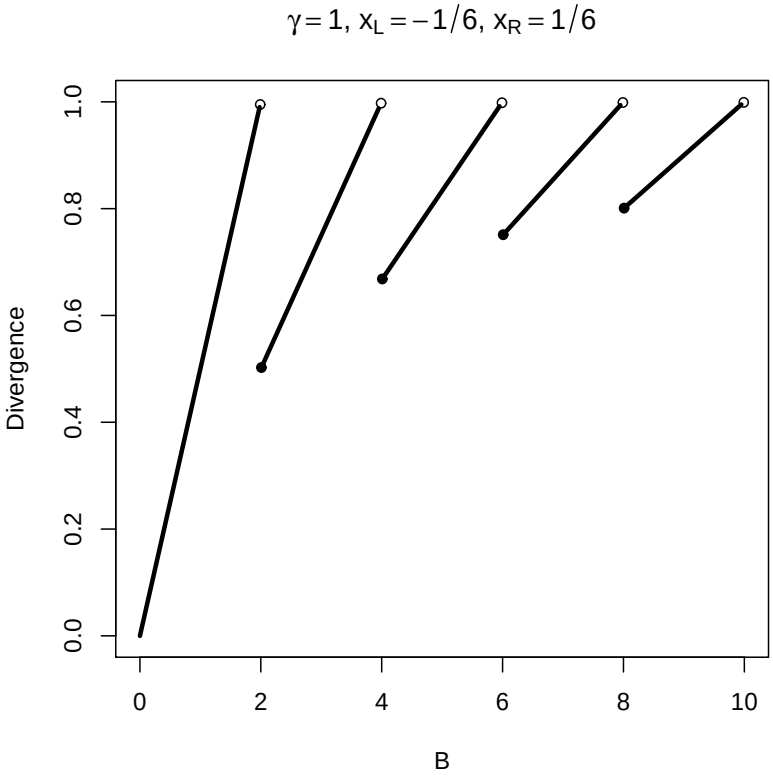

Figure 7: As the ratio of benefits to winning office over marginal costs of valence accumulation increases, platform divergence increases, except at a countable number of discontinuity points, where divergence decreases discontinuously.

up for the earlier increases in divergence.

We can also calculate whether there is a trend in the maximal level of divergence reached on each component. We can calculate the supremum of divergence on a component by taking the same value of B for which divergence has a discontinuous jump, but not increasing the value of n. At any such supremum, the level of divergence, then, is given by

$$\frac{B}{2(n-1)\gamma} = \frac{2(n-1)\gamma^2}{2(n-1)\gamma} = \gamma.$$

Thus, the supremum of divergence is a constant.

These relationships are illustrated in Figure 7.

6 Discussion

In this section we discuss the robustness of the model to alternative assumptions and situate our results in the existing literature.

6.1 Forms of Probabilistic Voting

Our model highlights the substantive importance of how one models uncertainty about voter preferences. Most of the applied literature employs the "stochastic partisanship" model, in which there is a stochastic shock to one of the candidate's valences. We, however, use the "stochastic preference" model, where there is uncertainty about the median voter's ideal point. With stochastic preferences, changes in polarization change the voter's responsiveness to increased valence accumulation, the key force at work in our model. With stochastic partisanship, no such interaction occurs. Thus, moving to stochastic partisanship would lead to two important changes in the model's predictions: (*i*) platform polarization would have no effect on valence accumulation[7] and (*ii*) there would be complete platform convergence in equilibrium.

To be more precise, consider a model in which there is no uncertainty about the voter's ideal point (which is fixed at x^*), but that before voting (but after platforms and valences are set) the voter's payoff for electing candidate L is changed by κ, a random variable distributed uniformly on $[-1, 1]$.

The voter votes for candidate L if and only if

$$\kappa \geq v_R - v_L + 2(x_R - x_L)\left(x^* - \frac{x_R + x_L}{2}\right).$$

Thus, the probability that the L candidate wins is

$$1 - \frac{1}{2}\left(v_R - v_L + 2(x_R - x_L)\left(x^* - \frac{x_R + x_L}{2}\right)\right).$$

In the valence subgame, the L candidate solves

$$\max_{v_L}\left(1 - \frac{1}{2}\left(v_R - v_L + 2(x_R - x_L)\left(x^* - \frac{x_R + x_L}{2}\right)\right)\right)B - v_L.$$

The marginal benefit of valence is $\frac{B}{2}$ and the marginal cost is 1. Polarization, thus, has no effect on valence choices. Moreover, if $B < 2$, the unique equilibrium has no valence accumulation (i.e., $v_L = v_R = 0$). In this event, the game reduces to standard Downsian competition and there is complete convergence of platforms. Alternatively, if $B > 2$, then there are only mixed strategy equilibria in the valence subgame and they are of exactly the form solved for in our model. Thus, the equilibrium payoff for candidate L in the valence

[7] Unless one also introduces policy oriented politicians, as in Herrera, Levine and Martinelli (2005).

subgame is the payoff if both candidates choose zero valence:

$$\left(1 - x^*(x_R - x_L) + \frac{x_R^2 - x_L^2}{2}\right) B,$$

which is clearly maximized at $x_L = x^*$. A similar argument follows for candidate R. Thus, even with positive valence accumulation, assuming a stochastic shock to valence, rather than uncertainty about the voter's ideal point, yields complete platform convergence.

6.2 Simultaneous Choice of Platforms and Valence

An important assumption of our model is that valence is chosen after platforms. Indeed, it might seem that this assumption is critical for our platform divergence result, since it is the anticipation of future valence competition that creates incentives for divergence. Thus, a natural question is the following: with simultaneous choice of platforms and valences does the equilibrium reduce to complete convergence?

The answer is no. While a complete characterization of equilibrium in the simultaneous move game is beyond the scope of this paper, we can demonstrate incentives for divergence when there is endogenous valence are robust to simultaneous moves.

Consider a game identical to the one we study, except that valences and platforms are chosen simultaneously. In any Nash equilibrium of this game, at least one candidate must choose a strategy that induces a non-degenerate lottery over platforms. Thus, there cannot be a convergent equilibrium.

To see why this is true, consider a strategy profile in which each candidate assigns probability one to some platform and these platforms are different from one another. For standard "Downsian" reasons, a candidate wants to deviate to a new strategy that holds her marginal over valences constant but moves her platform closer to the other candidate's.

This confirms that there cannot be pure platform choices when the platforms are diverged. However, we also need to show that pure converged platforms cannot be part of an equilibrium. We do this in two steps. First, consider a strategy profile with convergent platforms. If this strategy profile is an equilibrium, the candidates must be playing the uniform mixed strategy equilibrium over valences from Lemma 1. This implies that the candidates are getting expected payoffs of zero. Second, consider the deviation to a strategy where a candidate chooses zero valence with certainty and a platform that is slightly diverged (in the direction of the median voter's expected ideal point, if they were not already converged there). Here, the cost of valence is zero. However, the deviating candidate still wins with positive probability. This is because the non-deviating candidate assigns positive probability to very small amounts of valence, the platforms are close to each other, and there is probabilistic voting. Thus, the deviating player is now getting positive expected payoffs.

6.3 Other Mechanisms Linking Divergence and Endogenous Valence

The papers most closely related to this one are Zakharov (2005) and Eyster and Kittsteiner (2007). Zakharov (2005) studies a model similar to ours, but restricts attention to local equilibria. In work done simultaneously and independently Eyster and Kittsteiner (2007), like us, apply the logic of differentiation to soften competition to an electoral context. In their model, political parties that run candidates in a continuum of districts must choose a common platform for all candidates. Then each candidate has the option of tailoring the platform to her own district at some cost. They find that, for some parameter values, the parties choose divergent platforms so that the subsequent tailoring game will be less costly.

Although the basic intuition leading to divergence is the same in their and our papers, we view the contributions as complementary. One important difference is that in their model both the first and second stage choices for the politicians are about platforms whereas is our model the first stage is platforms and the second stage is valence. This has two interesting implications. First, they get divergence focusing solely on platform choices, whereas, for us, divergence depends on the introduction of an orthogonal dimension of competition. Second, in their model, the second stage cost is bounded by the fact that moving beyond the district's median voter is dominated. In our model, the only dominance consideration is that a candidate not choose a valence that costs more than the benefit from winning. Thus there is never convergence in our model, while they do find exact convergence for a sufficiently great benefit of winning election.

There are also a couple of smaller differences. First, our model works in a single district while, in their main model, it is crucial that there are many districts.[8] Second, the subgames in their model are essentially (potentially asymmetric) all-pay auctions with bid caps, while our subgames are essentially (potentially asymmetric) all-pay auctions with stochastic allocation rules. This difference leads to interesting differences in the equilibria.

Three other approaches also explore the relationship between endogenous valence and platform divergence, but based on significantly different intuitions.

Meirowitz (2004) studies a model in which the order of play is the opposite from ours—candidates first compete to accumulate valence and then choose platform locations. He finds that the platform choices are the same as those found by Aragones and Palfrey (2002) in a model with exogenous valence advantages.

Herrera, Levine and Martinelli (2005) study a model with the same order of play as our model. Their model differs from ours in two important ways—they have policy motivated

[8] In their conclusion, they mention an extension that works in a single district. In that extension, platforms are selected prior to an informative signal about the median voter's ideal point, with the possibility of modifying the platform after the signal. Again, this contrasts with our model, where both platform and valence decision are made with the same information about the voter.

politicians and they employ the stochastic partisanship model (i.e., uncertainty is due to a valence shock to one of the candidates). As a result of these two differences, they find that platform polarization and campaign spending covary positively in equilibrium (the opposite of what we find). This is because, as polarization increases, the value of winning election increases for a policy motivated politician.

Schofield (2003) and Carrillo and Castanheira (2006) have another set of intuitions. In these models politicians diverge because doing so increases valence. In Schofield (2003) it does so by motivating outside activists. In Carrillo and Castanheira (2006) the politicians themselves invest in valence and divergence in a previous stage makes the implicit promise to do so credible.

These various approaches to modeling endogenous platforms and endogenous valence lead to differing results about the relationship between these two choices and capture different intuitions. However, they also share certain important commonalities. In particular, like the earlier literature on platform choice with exogenous valence, they all confirm that, at least for some parameter values, allowing candidates to endogenously choose both platforms and valence leads to a robust prediction of platform divergence in elections.

A Proofs

A.1 Proof of Lemma 2

First we handle the case of $2\gamma(x_R - x_L) > B$. Assume that a candidate, say L, has $v_L > 0$ in the subgame. Then she must satisfy the first-order condition

$$\frac{\partial}{\partial v_L}\left(B\left(\frac{1}{2} + \frac{x_R + x_L}{2\gamma} + \frac{v_L - v_R}{2\gamma(x_R - x_L)}\right) - v_L\right) \geq 0.$$

(Any optimal choice that leads her to win with probability 1 must be the least costly such choice.) Taking the derivative gives

$$\frac{B}{2\gamma(x_R - x_L)} \geq 1,$$

which is clearly inconsistent with the hypothesis.

Note that mixing by the opponent makes $v_L > 0$ look even worse. Thus there is no equilibrium in which $v_L > 0$ with positive probability. By symmetry, there is no equilibrium in which $v_R > 0$ with positive probability.

Glicksberg's theorem tells us that there is an equilibrium, and the argument above implies it must be $v_L = v_R = 0$.

Second, when $2\gamma(x_R - x_L) = B$, there are many pure-strategy equilibria, one of which is $(v_l, v_R) = (0, 0)$. □

A.2 Proof of Lemma 3

Part 1

The payoff functions are continuous, so we have existence by Glicksberg's theorem.

Part 2

We start by showing that there is no pure strategy equilibrium in the subgame in which both candidates win with positive probability. If there were such an equilibrium, then the first-order conditions must hold with equality. But this is impossible, since the first order condition with equality is $\frac{B}{2\gamma(x_R - x_L)} = 1$.

Observe that this means at most one player can accumulate any valence in a pure-strategy equilibrium—one candidate must lose with probability 1, and that candidate optimizes by setting $v = 0$.

Next we show that there is no pure-strategy equilibrium in which both players choose $v = 0$. Again we argue by contradiction, so assume that the profile $(v_L, v_R) = (0, 0)$ is a

Nash equilibrium. Recall that there is no pure equilibrium in which both candidates win with positive probability, so one candidate wins for sure.

To get a contradiction, we need to show that one of the players has a profitable deviation. Since a player who wins with probability 1 at zero valence cannot possibly have a profitable deviation, we only need to look at deviations by the player who wins with probability 0. If $x_R^2 - x_L^2 > 0$, then this is player R, while if $x_R^2 - x_L^2 < 0$, then this is player L. (If $x_R^2 = x_L^2$, then the players each have probability 1/2 of winning when both choose 0 valence.)

So consider the case where $x_R^2 - x_L^2 < 0$. We introduce the notation $\alpha = 2\gamma(x_R - x_L)$. (Note that we are suppressing the functional dependencies.) Also, let $\epsilon = x_R^2 - x_L^2$.

Candidate L's best deviation is to the v that solves

$$\frac{1}{2} + \frac{\epsilon}{\alpha} + \frac{v}{\alpha} = 1,$$

or

$$v = \frac{\alpha}{2} - \epsilon.$$

To see that this is the best deviation, notice that a profitable deviation must involve a positive probability of winning, and the impossibility of the first order condition holding with equality means that the best deviation has probability 1 of winning. Clearly, this probability should be achieved at least cost, giving the condition. To be profitable, this deviation must lead to a greater payoff than $v = 0$. The deviation is profitable if

$$B - v = B - \frac{\alpha}{2} + \epsilon > 0,$$

or

$$B - \gamma(x_R - x_L) > (x_R^2 - x_L^2).$$

But our assumptions imply that the LHS is greater than zero while the RHS is less than zero. Thus L does have a profitable deviation.

A similar argument for the case of $x_R^2 - x_L^2 > 0$ shows that R has a profitable deviation. Thus at least one player always has a profitable deviation, contradicting the assumption that we were at an equilibrium.

Finally, we have to rule out equilibria in which the player who wins with probability 1 chooses $v > 0$. For concreteness, assume that this player is candidate L. As before, the only possibility for v_L is $v_L = \alpha/2 - \epsilon$. But at this v_L, candidate R wins with positive probability when she chooses v_R small and positive. Thus she can only choose 0 if the first-order condition

$$\frac{B}{2\gamma(x_R - x_L)} \leq 1$$

is satisfied, which is impossible.

Part 3

We prove the result by contradiction. So assume that there are platforms $x_L \neq x_R$ such that the subgame (x_L, x_R) has an equilibrium in which the strategies are given by cdfs (F_L, F_R) that are continuous and strictly increasing on their supports.

Let $\overline{v}_c = \sup \operatorname{supp} F_c$, the "top" of candidate c's support. In addition, let

$$v_c^* = \inf\{v \mid \Pr(c \text{ wins at } (v, \overline{v}_{-c})) > 0\}.$$

We can calculate these as follows:

$$\frac{1}{2} + \frac{\epsilon}{\alpha} + \frac{v_L^* - \overline{v}_R}{\alpha} = 0$$

implies

$$v_L^* = \overline{v}_R - \frac{\alpha}{2} - \epsilon;$$

and

$$\frac{1}{2} + \frac{\epsilon}{\alpha} + \frac{\overline{v}_L - v_R^*}{\alpha} = 1$$

implies

$$v_R^* = \overline{v}_L - \frac{\alpha}{2} + \epsilon.$$

We claim that equilibrium implies $\overline{v}_L \leq v_L^*$ and $\overline{v}_R \leq v_R^*$. From there, it's easy to deduce a contradiction: we have

$$\begin{aligned}
\overline{v}_L &\leq v_L^* \\
&= \overline{v}_R - \frac{\alpha}{2} - \epsilon \\
&\leq \left(\overline{v}_L - \frac{\alpha}{2} + \epsilon\right) - \frac{\alpha}{2} - \epsilon \\
&= \overline{v}_L - \frac{\alpha}{2},
\end{aligned}$$

a contradiction because $\alpha > 0$.

Finally, we prove the claim. (We do so only for candidate L; the proof for R is analogous.) If L chooses $v > v_L^*$, then her payoff against F_R is

$$B\left(\int_{v-\alpha/2+\epsilon}^{\overline{v}_R} \left(\frac{1}{2} + \frac{\epsilon}{\alpha} + \frac{v - \tilde{v}_R}{\alpha}\right) f_R(\tilde{v}_R)\, d\tilde{v}_R + F_R\left(v - \frac{\alpha}{2} + \epsilon\right)\right) - v.$$

If v is a best response, then it must satisfy the first-order condition

$$B\left(\int_{v-\alpha/2+\epsilon}^{\overline{v}_R} \frac{1}{\alpha} f_R(\tilde{v}_R)\, d\tilde{v}_R - f_R\left(v - \frac{\alpha}{2} + \epsilon\right) + f_R\left(v - \frac{\alpha}{2} + \epsilon\right)\right) - 1 = 0,$$

or

$$1 - F_R\left(v - \frac{\alpha}{2} + \epsilon\right) = \frac{\alpha}{B}.$$

But strict monotonicity of F_R implies that this equation has a unique solution, so there can be at most one best response for L greater than or equal to v_L^*. And since F_L is strictly increasing on its support, that means $\overline{v}_L \leq v_L^*$. □

A.3 Proof of Lemma 4

We've already seen that R's mixture must assign probability $\frac{\alpha}{B}$ to the interval $I(v_L, x_L, x_R)$. Since a non-overlapping profile assigns positive probability to exactly one point in that interval, it must assign that point probability exactly $\frac{\alpha}{B}$.

We've also seen that all of the probability not devoted to non-zero atoms is assigned to the valence of zero. At that point, the first order condition for L is

$$\Pr(v_R = 0)\frac{B}{\alpha} - 1 \leq 0,$$

so $\Pr(v_R = 0)$ must not exceed $\frac{\alpha}{B}$. (A similar argument holds for $\Pr(v_L = 0)$.) Thus we must create as many non-zero atoms as possible. The next Lemma implies that the probability assignment described in points 1–3 in fact uses up all of the "space" for non-zero atoms. It also implies the bounds in point 2.

Lemma 9

$$(1 - x\ T(1/x)) \leq x,$$

with equality if and only if $1/x$ is an integer. Furthermore, the expression is decreasing at every continuity point.

Proof Consider an x such that

$$\frac{1}{n+1} < x \leq \frac{1}{n}.$$

This holds if and only if

$$n \leq \frac{1}{x} < n + 1,$$

so $T(1/x) = n$ for all $x \in (1/(n+1), 1/n]$. Thus

$$(1 - x\ T(1/x)) = (1 - xn)$$

is decreasing in x on $(1/(n+1), 1/n]$. Finally, we have

$$x\ T(1/x) = xn \leq \frac{1}{n}n = 1$$

with equality if and only if $x = 1/n$, so the claimed inequality holds. □

A.4 Proof of Lemma 5

Note that, when valences are v_L and v_R, each candidate wins with positive probability only if $v_R \in I(v_L, x_L, x_R)$. (This follows from the construction of I.) This probability is

$$\beta^k = \frac{1}{2} + \frac{\epsilon}{\alpha} + \frac{v_L^k - v_R^k}{\alpha}.$$

Furthermore, L wins for sure if $v_R \leq \min I(v_L, x_L, x_R)$, and R wins for sure if $v_R \geq \max I(v_L, x_L, x_R)$. Thus we can write the payoff to L when she chooses valence v_L^k as

$$\pi_L^k = \sum_{j<k} Bp_R^j + Bp_R^k \beta^k - v_L^k. \tag{4}$$

To establish the first two points of the lemma, notice that in a mixed strategy equilibrium, the interior valences all must give equal expected payoffs. Then we need to choose v_L^2 and v_R^2 so that this common payoff is also the payoff to choosing $v_L^1 = v_R^1 = 0$. Again by construction of I, candidate c loses if he chooses zero valence and the other candidate chooses a positive amount of valence in $\mathcal{V}$. Thus, the payoff to candidate c of choosing zero valence is simply the probability the other player chooses zero valence times the probability candidate c wins when they both choose zero valence, times B. This payoff must be the same as the payoff from choosing the interior valence v_c^2. Using the fact (from Lemma 4) that $p_c^2 = \frac{\alpha}{B}$, this yields the following two equalities:

$$\begin{aligned} Bp_L^1 \beta^1 &= Bp_L^1 + Bp_L^2 \beta^2 - v_L^2 \\ &= Bp_L^1 + \alpha\beta^2 - v_L^2 \end{aligned}$$

and

$$\begin{aligned} Bp_R^1 \left(1 - \beta^1\right) &= Bp_R^1 + Bp_R^2 (1 - \beta^2) - v_R^2 \\ &= Bp_R^1 + \alpha(1 - \beta^2) - v_R^2. \end{aligned}$$

By the definition of β^2, we have

$$\beta^2 = \frac{1}{2} + \frac{x_R + x_L}{2\gamma} + \frac{v_L^2 - v_R^2}{\alpha}.$$

Further note that since $p_c^1 = 1 - n\frac{\alpha}{B}$, we have $p_L^1 = p_R^1 \equiv p^1$.

Rearrangement shows that these three equations with three unknowns have a unique solution in which

$$v_L^2 = Bp_L^1\beta^1 + \alpha(1 - \beta^1), \tag{5}$$

$$v_R^2 = Bp_R^1(1 - \beta^1) + \alpha\beta^1, \tag{6}$$

and

$$\beta^2 = \frac{Bp^1}{\alpha}(2\beta^1 - 1) + 1 - \beta^1. \tag{7}$$

To establish the third point, subtract π_L^k from π_L^{k+1} (see equation 4) to get

$$\begin{aligned} \pi_L^{k+1} - \pi_L^k &= \sum_{j<k+1} Bp_R^j + Bp_R^{k+1}\beta^{k+1} - v_L^{k+1} - \left(\sum_{j<k} Bp_R^j + Bp_R^k\beta^k - v_L^k\right) \\ &= Bp_R^{k+1}\beta^{k+1} + Bp_R^k(1 - \beta^k) - (v_L^{k+1} - v_L^k). \end{aligned}$$

In a mixed strategy equilibrium, this difference must be zero. Thus, if $k > 1$, we have

$$\alpha(\beta^{k+1} + 1 - \beta^k) = v_L^{k+1} - v_L^k.$$

Rearrangement now gives the result. A similar calculation works for R.

A.5 Proof of Proposition 1

We prove existence by constructing an equilibrium. After doing so, we prove that all non-overlapping provide the same equilibrium payoffs (as specified in point two of the proposition).

A.5.1 Existence

Again, we use the notation $\alpha = 2\gamma(x_R - x_L)$ and $\epsilon = x_R^2 - x_L^2$. Then the probability that L wins is

$$\frac{1}{\alpha}\left(\frac{\alpha}{2} + \epsilon + v_L - v_R\right),$$

assuming that this leads to an interior probability.

Let $n = T(B/\alpha) + 1$, $v_c^1 = 0$, and $v_c^{k+1} - v_c^k = \alpha$ for all $k \geq 1$. Candidate c chooses a valence from the finite set

$$\mathcal{V}_c = \{v_c^1, v_c^2, \ldots, v_c^n\}$$

according to the probability measure

$$(p_c^1, p_c^2, \ldots, p_c^n) = (1 - (\alpha/B)\, T(B/\alpha), \alpha/B, \ldots, \alpha/B)\,,$$

where the equality is established by Lemma 4.

Further, by Lemma 5, the fact that $v_c^{k+1} - v_c^k = \alpha$ implies that $\beta^k = \beta^{k+1} \equiv \beta$ for all $k > 1$.

Now from equations (5)-(7) in the proof of Lemma 5, we have

$$\begin{aligned} v_R^2 &= Bp^1(1-\beta^1) + \alpha\beta^1 \\ v_L^2 &= Bp^1\beta^1 + \alpha(1-\beta^1) \\ \beta &= \frac{Bp^1}{\alpha}(2\beta^1 - 1) + 1 - \beta^1. \end{aligned}$$

To constitute an mixed strategy equilibrium, several other conditions must be satisfied. First, β must be a probability, else our specification of the payoffs is wrong. Second, the v^2's must be far enough from zero that they do not upset the first order conditions for zero valence. This requires

$$\frac{1}{2} + \frac{\epsilon}{\alpha} - \frac{v_R^2}{\alpha} \leq 0 \quad \Rightarrow \quad v_R^2 \geq \alpha\beta^1$$

and

$$\frac{1}{2} + \frac{\epsilon}{\alpha} + \frac{v_L^2}{\alpha} \geq 1 \quad \Rightarrow \quad v_L^2 \geq \alpha(1-\beta^1).$$

The first says that if L chooses 0 and R chooses v_R^2, then L loses for sure, while the second says that if L chooses v_L^2 and R chooses 0, then L wins for sure. Third, the strategies must not imply anyone is playing a strictly dominated strategy, so we need v_L^n and v_R^n to not exceed B.

Thus we look for a solution (β, v_L^2, v_R^2) to the system

$$\begin{aligned}
\beta &= \frac{1}{2} + \frac{\epsilon}{\alpha} + \frac{v_L^k - v_R^k}{\alpha} && (8)\\
v_R^2 &= Bp^1(1-\beta^1) + \alpha\beta^1 && (9)\\
v_L^2 &= Bp^1\beta^1 + \alpha(1-\beta^1) && (10)\\
\beta &\in [0,1] && (11)\\
v_R^2 &\geq \alpha\beta^1 && (12)\\
v_L^2 &\geq \alpha(1-\beta^1) && (13)\\
B &\geq v_R^2 + (n-2)\alpha && (14)\\
B &\geq v_L^2 + (n-2)\alpha. && (15)
\end{aligned}$$

Lemma 10 *If (β, v_L^2, v_R^2) solve 8–15, then the implied mixed strategies are an equilibrium.*

Proof Each candidate is indifferent across all of her choices by construction, and the first order conditions guarantee that she cannot do better from deviations in the "scope" of the played strategies. Thus the only deviations to worry about are those where

$$\frac{1}{2} + \frac{\epsilon}{\alpha} + \frac{v - v_R^n}{\alpha} \geq 1$$

or $v > 0$ and

$$\frac{1}{2} + \frac{\epsilon}{\alpha} + \frac{v - v_R^2}{\alpha} \leq 0.$$

(Clearly, the focus on L is without loss of generality.) But such choices are dominated by choices that are covered by the previous arguments. □

Now all we have to do is check the inequalities. First we show that $v_R^2 \geq \alpha\beta^1$. Substitute in the solution for v_R^2 to see that this inequality is equivalent to

$$Bp^1(1-\beta^1) + \alpha\beta^1 \geq \alpha\beta^1,$$

which is clearly true. A similar argument shows that $v_L^2 \geq \alpha(1-\beta^1)$.

Second we check that β is between 0 and 1. Recall that

$$\beta = \frac{Bp^1}{\alpha}(2\beta^1 - 1) + 1 - \beta^1.$$

There are three subcases to consider:

1. $\beta^1 > 1/2$. Here the first term is positive. Since $p^1 \in [0, \alpha/B)$ we have

$$1 - \beta^1 = \inf_{p^1} \frac{Bp^1}{\alpha}(2\beta^1 - 1) + 1 - \beta^1 \leq \beta \leq \sup_{p^1} \frac{Bp^1}{\alpha}(2\beta^1 - 1) + 1 - \beta^1 = \beta^1,$$

   and β is a probability.

2. $\beta^1 = 1/2$. Here we have $\beta = 1 - \beta^1$, a probability.

3. $\beta^1 < 1/2$. Here the first term is negative. Thus we have

$$\beta^1 = \inf_{p^1} \frac{Bp^1}{\alpha}(2\beta^1 - 1) + 1 - \beta^1 \leq \beta \leq \sup_{p^1} \frac{Bp^1}{\alpha}(2\beta^1 - 1) + 1 - \beta^1 = 1 - \beta^1,$$

   and β is a probability.

Third we check the constraint that no valence in the support of either mixed-strategy exceeds B. Clearly it suffices to check that $v_L^n \leq B$ and $v_R^n \leq B$. We have

$$v_R^n = v_R^2 + (n-2)\alpha = Bp^1(1 - \beta^1) + \alpha\beta^1 + (n-2)\alpha.$$

Lemma 9 says $p^1 \leq \alpha/B$, so we have

$$\begin{aligned} Bp^1(1 - \beta^1) + \alpha\beta^1 + (n-2)\alpha &\leq \alpha + (n-2)\alpha \\ &= \alpha(n-1) \\ &\leq B, \end{aligned}$$

where the last inequality follows from the definition of n. Next, we have

$$v_L^n = v_L^2 + (n-2)\alpha = Bp^1\beta^1 + \alpha(1 - \beta^1) + (n-2)\alpha/B.$$

Now a similar argument gives us $v_L^n \leq B$. This establishes equilibrium existence.

A.5.2 Equilibrium Payoffs

In any non-overlapping equilibrium, each candidate plays zero with positive probability. By the indifference property of mixed strategy equilibria, the equilibrium payoff must be the payoff to playing zero. Non-overlappingness implies that this payoff is just the probability that the other player plays zero times the probability of victory given no valence is accumulated at all. And that quantity is pinned down independently of the details of the rest of the equilibrium.

In particular, the probability that either party chooses zero valence is

$$\begin{aligned} p^1 &= 1 - n\frac{\alpha}{B} \\ &= 1 - \frac{\alpha}{B} T\left(\frac{B}{\alpha}\right) \\ &= 1 - \frac{2\gamma(x_R - x_L)}{B} T\left(\frac{B}{2\gamma(x_R - x_L)}\right). \end{aligned}$$

The probability that party L wins, given that both candidates choose valence equal to zero is

$$\begin{aligned} \beta_L^1 &= \frac{1}{2} + \frac{\epsilon}{\alpha} \\ &= \frac{1}{2} + \frac{x_R + x_L}{2\gamma}. \end{aligned}$$

The probability that party R wins when both choose zero valence is $1 - \beta_L^1$. Finally, the payoff off victory given no valence is simply B.

Combining these, the expected payoffs from choosing zero valence (and, thus, the expected equilibrium payoffs) are

$$\pi_L(x_L, x_R) = B\left(\frac{1}{2} + \frac{x_R + x_L}{2\gamma}\right)\left(1 - \frac{2\gamma(x_R - x_L)}{B}\, T\left(\frac{B}{2\gamma(x_R - x_L)}\right)\right)$$

to candidate L and

$$\pi_R(x_L, x_R) = B\left(\frac{1}{2} - \frac{x_R + x_L}{2\gamma}\right)\left(1 - \frac{2\gamma(x_R - x_L)}{B}\, T\left(\frac{B}{2\gamma(x_R - x_L)}\right)\right).$$

to candidate R, establishing the result.

A.6 Proof of Proposition 2

Recall from Lemma 5 that the valences in the supports of the equilibrium mixtures satisfy

$$v_L^{k+1} - v_L^k = \alpha + \alpha(\beta^{k+1} - \beta^k)$$

and

$$v_R^{k+1} - v_R^k = \alpha - \alpha(\beta^{k+1} - \beta^k)$$

with the initial conditions

$$v_L^1 = v_R^1 = 0,$$

$$v_L^2 = Bp^1\beta^1 + (1 - \beta^1)\alpha,$$

and

$$v_R^2 = Bp^1(1-\beta^1) + \beta^1\alpha.$$

Because the platforms are symmetric, $\beta^1 = 1/2$.

The solutions to these difference equations are

$$v_k^L = v_L^2 + (k-2)\alpha + \delta_k$$

and

$$v_k^R = v_R^2 + (k-2)\alpha - \delta_k$$

for some common sequence $\{\delta_k\}$. Thus we have

$$\mathbb{E}(v^L) = \mathbb{E}(v_L^2 + (k-2)\alpha) + \mathbb{E}(\delta_k)$$

and

$$\mathbb{E}(v^R) = \mathbb{E}(v_R^2 + (k-2)\alpha) - \mathbb{E}(\delta_k).$$

Then independence of the players' mixtures and the fact that the δ have the same distribution across players imply that

$$\mathbb{E}(v^L + v^R) = 2\mathbb{E}(v_L^2 + (k-2)\alpha).$$

We complete the proof by calculating:

$$\begin{aligned}
\mathbb{E}(v_L^2 + (k-2)\alpha) &= \sum_{k=2}^{n} p^k \left(v_L^2 + (k-2)\alpha\right) \\
&= \frac{\alpha}{B}(n-1)v_L^2 + \frac{\alpha^2}{B}\sum_{k=2}^{n}(k-2) \\
&= (n-1)\frac{\alpha}{B}\left(\frac{1}{2}Bp^1 + \frac{1}{2}\alpha + \frac{1}{2}\alpha(n-2)\right) \\
&= T(B/\alpha)\frac{\alpha}{B}\left(\frac{1}{2}B\left(1 - \frac{\alpha}{B}T(B/\alpha)\right) + \frac{1}{2}\alpha + \frac{1}{2}\alpha(T(B/\alpha)-1)\right) \\
&= T(B/\alpha)\frac{\alpha}{B}\left(\frac{1}{2}B - \frac{1}{2}\alpha T(B/\alpha) + \frac{1}{2}\alpha + \frac{1}{2}\alpha T(B/\alpha) - \frac{1}{2}\alpha\right) \\
&= \frac{1}{2}\alpha T\left(\frac{B}{\alpha}\right).
\end{aligned}$$

A.7 Proof of Lemma 7

Immediate from Lemma 9.

A.8 Proof of Lemma 8

Consider the stated profile for $n > 0$. Accumulation behavior is a non-overlapping equilibrium for every subgame by Lemmata 1, 2, and Proposition 1, while any selection of non-overlapping equilibria in the accumulation subgames yields the same payoffs as the one specified by Proposition 1. Voter behavior is an equilibrium for every subgame if and only if condition 3 holds by the discussion surrounding equation (2).

Lemma 6 says implies that, to have a non-overlapping equilibrium, the distance between the platforms, $x_R - x_L$, must make the ratio $\frac{B}{2\gamma(x_R - x_L)}$ an integer. Here we have

$$\frac{B}{2\gamma\left(\frac{B}{2n\gamma}\right)} = n,$$

and the condition is satisfied. And by Lemma 6, the distance between the platforms, $x_R - x_L$, must make the ratio $\frac{B}{2\gamma(x_R - x_L)}$ an integer, say n. Thus the platforms must be $\frac{B}{2\gamma n}$ apart. And since we are restricting attention to symmetric equilibria, the platforms must be

$$x_L = -\frac{B}{4\gamma n} \quad \text{and} \quad x_R = \frac{B}{4\gamma n}.$$

So far, we have shown that any symmetric non-overlapping equilibrium has the form of 1–3, and that, for any n, play according to 2 and 3 is an equilibrium in the proper subgames. Thus all we need to establish is that the platforms are in equilibrium if and only if the n satisfies the bounds.

By symmetry, it suffices to check for profitable deviations for just one candidate, say L. Lemma 6 implies that we only need to rule out deviations that make $\frac{B}{2\gamma(x_R - x_L)}$ and integer, and Lemma 7 implies that we only need to check deviations to "adjacent" platforms: candidate L must not want to deviate in the direction of candidate R's location to the platform x'_L that satisfies $B/2\gamma(x_R - x'_L) = n + 1$, or

$$x'_L = -B\frac{n-1}{4n(n+1)\gamma}. \tag{16}$$

Similarly, she must not want to deviate away from candidate R's location to the platform x''_L that satisfies $B/2\gamma(x_R - x''_L) = n - 1$, or

$$x''_L = -B\frac{n+1}{4n(n-1)\gamma}. \tag{17}$$

This gives two conditions to check:

1. $f\left(-\frac{B}{4n\gamma}, \frac{B}{4n\gamma}\right) \geq f\left(\frac{B-n}{4n(n+1)\gamma}, \frac{B}{4n\gamma}\right)$, and

2. $f\left(-\frac{B}{4n\gamma}, \frac{B}{4n\gamma}\right) \geq f\left(\frac{-B-n}{4n(n-1)\gamma}, \frac{B}{4n\gamma}\right)$.

We consider each of these in turn.

1. Expanding an earlier expression for f, we need

$$\gamma x_R - \gamma x_L + x_R^2 - x_L^2 \geq \gamma x_R - \gamma x_L' + x_R^2 - (x_L')^2,$$

or

$$\gamma(x_L' - x_L) + (x_L')^2 - x_L^2 \geq 0.$$

Substitute to get

$$\gamma B\left(\frac{1-n}{4n(n+1)\gamma} + \frac{1}{4n\gamma}\right) + B^2(\frac{1-n}{4n(n+1)\gamma})^2 - B^2\frac{1}{16n^2\gamma^2} \geq 0,$$

which simplifies to

$$\frac{1}{2}\frac{B}{\gamma^2} - 1 \leq n.$$

2. This time we need

$$\gamma(x_L'' - x_L) + (x_L'')^2 - x_L^2 \geq 0.$$

Substitute to get

$$\gamma B\left(\frac{-1-n}{4n(n-1)\gamma} + \frac{1}{4n\gamma}\right) + B^2\left(\frac{-1-n}{4n(n-1)\gamma}\right)^2 - B^2\frac{1}{16n^2\gamma^2} \geq 0,$$

which simplifies to

$$n \leq \frac{1}{2}\frac{B}{\gamma^2} + 1.$$

□

References

Anderson, Simon P. 1988. "Equilibrium Existence in the Linear Model of Spatial Competition." *Economica* 55(220):479–491.

Anderson, Simon P. and Jacob K. Goeree. 1997. "Location, Location, Location." *Journal of Economic Theory* 77:102–127.

Ansolabehere, Stephen and James M. Snyder, Jr. 2000. "Valence Politics and Equilibrium in Spatial Election Models." *Public Choice* 103:327–336.

Aragones, Enriqueta and Thomas R. Palfrey. 2002. "Mixed Equilibrium in a Downsian Model with a Favored Candidate." *Journal of Economic Theory* 103:131–161.

Ashworth, Scott and Ethan Bueno de Mesquita. 2006. "Delivering the Goods: Legislative Particularism in Different Electoral and Institutional Settings." *Journal of Politics* 68(1):169–179.

Ashworth, Scott and Ethan Bueno de Mesquita. forthcoming. "Informative Party Labels with Institutional and Electoral Variation." *Journal of Theoretical Politics* .

Carrillo, Juan D. and Micael Castanheira. 2006. "Information and Strategic Political Polarization." USC typescript.

Che, Yeon-Koo and Ian Gale. 2000. "Difference-Form Contests and the Robustness of All-Pay Auctions." *Games and Economic Behavior* 30(1):22–43.

d'Aspremont, C., J. Jaskold Gabszewicz and J.-F. Thisse. 1979. "On Hotelling's 'Stability in Competition'." *Econometrica* 47(5):1145–1150.

Dickson, Eric S. and Kenneth Scheve. 2006. "Social Identity, Political Speech, and Electoral Competition." *Journal of Theoretical Politics* 18(1):5–39.

Downs, Anthony. 1957. *An Economic Theory of Democracy.* New York: Harper and Row.

Duggan, John. 2005. "A Survey of Equilibrium Analysis in Spatial Models of Elections." Unpublished manuscript.

Erikson, Robert S. and Thomas R. Palfrey. 2000. "Equilibria in Campaign Spending Games: Theory and Data." *American Political Science Review* 94(3):595–609.

Eyster, Erik and Thomas Kittsteiner. 2007. "Party platforms in electoral competition with heterogeneous constituencies." *Theoretical Economics* 2(1):41–70.

Groseclose, Timothy J. 2001. "A Model of Candidate Location When One Voter Has a Valence Advantage." *American Journal of Political Science* 45(4):862–886.

Herrera, Helios, David K. Levine and César Martinelli. 2005. "Policy Platforms, Campaign Spending and Voter Participation." Unpublished manuscript.

Londregan, John and Thomas Romer. 1993. Polarization, Incumbency, and the Personal Vote. In *Political Economy: Institutions, Competition, and Representation*, ed. William A. Barnett, Melvin J. Hinich and Norman J. Schofield. Cambridge: Cambridge University Press pp. 355–377.

Meagher, Kieron J. and Klaus G. Zauner. 2004. "Product Differentiation and Location Decisions Under Demand Uncertainty." *Journal of Economic Theory* 117:201–216.

Meirowitz, Adam. 2004. "Models of Electoral Campaigns." Unpublished paper.

Morton, Rebecca B. and Roger B. Myerson. 1992. "Decisiveness of Contributors' Perceptions in Elections." Unpublished manuscript.

Osborne, Martin J. and Carolyn Pitchik. 1987. "Equilibrium in Hotelling's Model of Spatial Competition." *Econometrica* 55:911–922.

Polo, M. 1998. "Electoral Competition and Political Rents." IGIER working paper n. 144.

Schofield, Norman J. 2003. "Valence Competition in the Spatial Stochastic Model." *The Journal of Theoretical Politics* 15(4):371–383.

Schofield, Norman J. Forthcoming. "The Mean Voter Theorem: Necessary and Sufficient Conditions for Convergent Equilibrium." *Review of Economic Studies* .

Snyder, James M. and Michael M. Ting. 2002. "An Informational Rationale for Political Parties." *American Journal of Political Science* 46(1):90–110.

Stokes, Donald E. 1963. "Spatial Models of Party Competition." *American Political Science Review* 57:368–377.

Tirole, Jean. 1988. *The Theory of Industrial Organization.* Cambridge: MIT Press.

Zakharov, Alexei. 2005. "Candidate Location and Endogenous Valence." Unpublished paper.