On the Optimal Timing of Benefits with Heterogeneous Workers and Human Capital Depreciation[*]

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Abstract

This paper studies the optimal timing of unemployment insurance subsidies in a McCall search model. Risk-averse workers sequentially sample random job opportunities. Our model distinguishes unemployment subsidies from consumption during unemployment by allowing workers to save and borrow freely. When the insurance agency faces a group of homogeneous workers solving stationary search problems, the optimal subsidies are independent of unemployment duration. In contrast, when workers are heterogeneous or when human capital depreciates during the spell, the optimal subsidy is no longer constant. We explore the main determinants of the shape of the optimal subsidy schedule, isolating forces for subsidies to optimally rise or fall with duration.

1. Introduction

When a homogeneous worker faces a stationary search problem, the optimal unemployment insurance subsidy is independent of the worker's unemployment duration (Shimer and Werning, 2005). In this paper we examine how relaxing these assumptions affects the optimal timing of unemployment subsidies. In particular, we model an unemployment insurance agency which sets a schedule of unemployment subsidies as a function of unemployment duration and faces either: (i) a group of homogeneous workers whose human capital depreciates during unemployment; or (ii) a group of heterogeneous workers.

Both scenarios are empirically relevant and represent a significant departure from previous work on optimal unemployment insurance, which focuses on the optimal contract for homogeneous workers with stationary search problems (e.g. Shavell and Weiss, 1979; Hopenhayn and Nicolini, 1997). Our work also differs from most research on optimal unemployment insurance by distinguishing unemployment consumption from unemployment subsidies. We allow workers to borrow and save

using risk free bonds. In Shimer and Werning (2005) we argued that this is crucial for understanding the design of unemployment insurance policy: a constant subsidy is optimal, but unemployed workers choose a declining path for their consumption. Intuitively, with homogeneous workers facing stationary search problems, constant subsidies are optimal because the tradeoff between insurance and incentives does not change over time.

This paper shows that the optimal subsidy schedule is not flat when we move away from the benchmark with identical workers facing stationary search problems. When workers' job opportunities deteriorate during the spell, or when the pool of unemployed workers shifts from one type to another, the tradeoff between insurance and incentives changes during the spell—and subsidies change with it. We provide a simple and tractable framework for exploring the main determinants of the optimal timing of subsidies. In particular, our model is well suited for isolating the impact of human capital depreciation and heterogeneity on the timing of subsidies, precisely because subsidies are constant in the version of our model without these features.

Throughout this paper, we focus on workers with constant absolute risk aver-

sion (CARA) preferences, since the absence of wealth effects leads to particularly clean results. In the benchmark model with neither human capital depreciation nor heterogeneity, optimal unemployment insurance subsidies are constant with CARA preferences and increase slowly with unemployment duration when workers have constant relative risk aversion (CRRA) preferences. Again, the fact that subsides are constant in the benchmark model is an important justification for the CARA specification.

We show that the optimal time-varying path of subsidies depends on the form of human capital depreciation or heterogeneity, and that there are forces for increasing or decreasing subsidy schedules. The main lesson we derive is that optimal subsidies tend to shift over the spell in the direction suggested by simple comparative statics. For instance, if a worker with constant but lower human capital merits a lower constant subsidy in a stationary environment, then when human capital depreciates over the spell, subsidies should fall with unemployment duration. Similarly, if workers are heterogeneous, subsidies fall if the composition of the unemployment pool shifts towards those workers who would merit a lower constant subsidy if we

could have, fictitiously, costlessly separated them.

In addition to this general finding, some interesting particular conclusions emerge from our numerical explorations. We consider two forms of human capital depreciation. In the first, job opportunities arrive at a constant rate but the wage distribution they are drawn from deteriorates in a steady and parallel fashion. This case captures skill depreciation and is close to that in Ljungqvist and Sargent (1998). The second form of depreciation has workers sampling from the same distribution of wages, but the arrival rate of opportunities falls over time. That is, search frictions rise during the unemployment spell. One interpretation is that workers become increasingly detached from the labor market as they initially exhaust their nearest sources for jobs and turn to remoter options. For concreteness, we call the first form of depreciation *skill depreciation* and the second form *search depreciation*.

In our skill depreciation exercises we find decreasing optimal unemployment insurance subsidies. Constant subsidies would give the long-term unemployed a higher replacement ratio relative to their potential wages and induce these workers to become overly picky, or even drop out, as stressed by Ljungqvist and Sargent

(1998). In our exercises, the declining wage opportunities lead the reservation wage to decline steadily during unemployment. However, subsidies decline even faster: the ratio of the unemployment insurance subsidy to the reservation wage is also decreasing. Interestingly, we also find that skill depreciation creates a force for larger initial unemployment subsidy levels. Intuitively, the shocks to workers' permanent income from remaining unemployed for an additional week, which we seek to insure, is larger because they are no longer simply the missed current earnings, but also include the lower future earnings.

In contrast, we find that search depreciation creates a force for rising subsidies. Intuitively, unfortunate workers who remain unemployed for a long time have lower arrival rates of offers and, therefore, demand more insurance to deal with their heightened duration risk. Long-term unemployed workers have lower exit rates from unemployment, but not because they become choosier. Indeed, in our exercises we find declining reservation wages during the spell. The reason for the lower employment rate is that they receive fewer job offers. Since the moral hazard problem becomes less severe, but risks loom greater, this leads to rising insurance

subsidies.

In our exercises with heterogeneity we also find forces for increasing or decreasing subsidies. Roughly speaking, subsidies fall if the pool of unemployed shifts over time towards workers that would have required lower constant subsidies had the agency faced them in isolation. For example, suppose workers differ in their value of leisure. Workers who value leisure more have a higher reservation wage and longer unemployment duration but less need for unemployment insurance. In this case, subsidies tend to decline during an unemployment spell. Conversely, if workers differ in the variance of their wage draws, those who face more idiosyncratic uncertainty have a higher option value of search, a higher reservation wage, longer unemployment duration, and a greater need for unemployment insurance. Benefits tend to rise during an unemployment spell.

In both cases, we find that optimal subsidies tend to overshoot their long-run target. To be concrete, suppose individuals differ in their value of leisure. If types could be separated, optimal subsidies would be constant for each type and higher for workers who value leisure less. A common decreasing subsidy schedule partially

imitates this desirable but unattainable property. The reason is that with any common subsidy schedule, unemployment duration is longer for workers who have a higher value of leisure and so the expected present value of per-period subsidies is lower for such workers when the subsidy schedule is decreasing with unemployment duration.

Put differently, the model predicts that optimal unemployment subsidies for the long-term unemployed should be low both because the long-term unemployed mostly have a high value of leisure and because this provides better a tradeoff between insurance and incentives for workers in the early stages of an unemployment spell. The second piece of the argument points to a time consistency problem: if it were possible to reset the subsidy, the insurance agency would like to raise unemployment subsidies. By constraining itself from doing that, it is possible to provide better incentives early in the unemployment spell.

Throughout this paper, we focus on the optimal timing of unemployment subsidies rather than unemployment benefits and taxes. This is because of Ricardian equivalence. When workers can borrow and lend, many unemployment benefit and

tax schedules are equivalent in the sense that they do not affect workers' budget sets and hence do not affect their behavior. However, the model uniquely determines the optimal timing of unemployment subsidies, the present value of transfers from the unemployment insurance agency if the worker remains unemployed for one additional period.

Our model incorporates elements present in various positive models of unemployment and the effects of unemployment insurance. Ljungqvist and Sargent (1998) emphasize human capital depreciation of unemployed workers to explain higher European unemployment. They model skill loss as stochastic, so their story actually also combines elements of heterogeneity. In particular, during 'tranquil' times human capital depreciates steadily during unemployment generating unimportant amounts of heterogeneity among the unemployed. In contrast, during 'turbulent' times a fraction of workers lose skills immediately at the moment they are laid off, generating significant amounts of heterogeneity.[1]

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[1] Ljungqvist and Sargent (1998, pg. 548) conclude that "during tranquil times, the depreciation of skills during spells of unemployment [...] is simply too slow to have much effect on the amount

Our paper departs in important ways from existing normative analyses that focuses on homogenous workers facing stationary search problems. In these contexts, Shavell and Weiss (1979) and Hopenhayn and Nicolini (1997) showed that consumption should decline during unemployment. Although these results are often interpreted as a prescription for declining unemployment subsidies, Werning (2002) and Shimer and Werning (2005) highlight the importance of distinguishing consumption from unemployment subsidies and suggest a different interpretation: workers collect constant, or near constant, unemployment benefits but choose to have consumption declining as they draw down their assets, by the usual permanent-income reasoning, to smooth their consumption. Pavoni (2003) and Violante and Pavoni (2005) are two papers that study optimal insurance in environments with human capital depreciation. However, these papers do not focus on the optimal timing of subsidies because they do not attempt to distinguish consumption from subsidies, and they focus on the latter by assuming that the unemployment

of long-term unemployed. The primary cause of long-term unemployment in our turbulent times is the instantaneous loss of skills at layoffs. Our probabilistic specification of this instantaneous loss creates heterogeneity among laid-off workers having the same past earnings."

insurance agency can control consumption.

The rest of the paper is organized into four sections. Section 2 describes the model. Section 3 characterizes the worker's search problem for any given unemployment insurance schedule. Section 4 describes optimal unemployment insurance policy. Section 5 studies homogeneous workers who face skill and search depreciation during unemployment. Section 6 turns to the case of heterogeneity. Section 7 contains our conclusions.

2. The Model

We adapt the model from Shimer and Werning (2005, 2006). These papers provide a tractable version of a McCall (1970) search problem enhanced to incorporate risk-averse workers that can save and borrow freely. Time is continuous and infinite $t \in [0, \infty)$. Workers seek to maximize expected discounted utility

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c(t)) dt \tag{1}$$

where $c(t)$ is consumption. We work with Constant Absolute Risk Aversion (CARA) utility functions: $u(c) = -e^{-\gamma c}$ with $c \in \mathbb{R}$. This assumption allows us to solve the model in closed form.[2]

Unemployed workers sample job opportunities at Poisson arrival rates $\lambda(t)$. Jobs are distinguished by their wage w drawn from a distribution $F(w, t)$. This introduces two potential forms of human capital depreciation. We assume jobs last forever.

Budget Constraints. Workers can save and borrow at the market interest r. Their budget constraints are

$$\dot{a}(t) = ra(t) + y(t) - c(t), \tag{2}$$

where $a(t)$ are assets and $y(t)$ represents current non-interest income: it equals the unemployment subsidy, $B(t)$, during unemployment and the wage, $w(t)$, during employment. Initial assets $a(0) = a_0$ are given, although there may be initial lump-

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[2]Shimer and Werning (2005) verify that CARA preferences provide a good benchmark: the numerical solution with CRRA preferences is very close to the CARA closed form solution.

sum taxes and transfers, as we discuss further below. In addition workers must satisfy the No-Ponzi condition $\lim_{t\to\infty} e^{-rt}a(t) \geq 0$.

Heterogeneity. To capture worker heterogeneity we assume that there are N types indexed by $n = 1, 2, \ldots, N$ and index the primitives of the worker's search problem by the type, $\lambda_n(t)$, $F_n(w, t)$, $u_n(c)$, γ_n, and ρ_n.

Unemployment Insurance Policy. In this paper we are interested in the optimal timing of unemployment insurance subsidies, and not in larger, more comprehensive, social welfare reforms. This motivates the policy problem we consider, which is to select a schedule of unemployment subsidies $\{B(t)\}_{t\geq 0}$ that stipulates the subsidy $B(t)$ received by a worker that remains unemployed at t. Subsidies can be conditioned only on unemployment duration.[3]

    The optimal unemployment insurance policy problem we study is to find the best such schedule of subsidies. For the case with heterogeneous workers one must, in general, specify a welfare criterion. However, to avoid redistributional concerns

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[3]We do not consider, for instance, menus of schedules that may self-select and separate worker types; likewise we do not consider constraining workers access to savings.

we allow initial lump-sum transfers between workers types. These transfers are equivalent to redistributions in terms of initial assets a_0. It turns out that, with CARA preferences, the optimal schedule $\{B(t)\}_{t \geq 0}$ is then uniquely pinned down: all Pareto efficient allocations can be achieved with the same schedule by varying the initial lump-sum transfers (initial assets) between workers. Thus, we do not need to specify any particular welfare criterion and our analysis characterizes all Pareto efficient schedules.

3. Worker Behavior

In this section we characterize the behavior of a single unemployed type n worker confronted with any subsidy schedule $\{B(t)\}_{t \geq 0}$.

For any job-acceptance policy, workers solve a standard consumption-savings subproblem, maximizing utility in equation (1) subject to the budget constraint equation (2) and the No-Ponzi condition. As is well known, the solution to any

consumption-savings problem must satisfy the usual intertemporal Euler equation

$$u_n{}'(\tilde{c}_t) = e^{-(\rho_n - r)s}\mathbb{E}_t[u_n{}'(\tilde{c}_{t+s})],$$

where $\{\tilde{c}_t\}$ is the optimal stochastic process for consumption. With CARA preferences $u_n{}'(c) = \gamma_n u_n(c)$, so this implies

$$u_n(\tilde{c}_t) = e^{-(\rho_n - r)s}\mathbb{E}_t[u_n(\tilde{c}_{t+s})].$$

It follows that

$$\int_0^\infty e^{-\rho_n s}\mathbb{E}_t[u(\tilde{c}_{t+s})]ds = \frac{u(\tilde{c}_t)}{r}$$

which conveniently relates lifetime utility to current consumption.

Let $V_n(t)$ and $c(t)$ represent the lifetime utility and consumption of an unemployed worker, respectively, after duration t; note that both are deterministic

functions of t. Then applying the argument above implies that

$$V_n(t) = \frac{u_n(c(t))}{r}. \tag{3}$$

Unemployed workers wait around for job offers. An unemployed worker accepts a job offer if the wage is higher than the wage $\bar{w}_n(t)$ which makes her indifferent. Since a worker with current assets $a(t)$ and a job paying $\bar{w}_n(t)$ forever consumes $ra(t) + \bar{w}_n(t) + (\rho_n - r)/\gamma_n r$,

$$V_n(t) = \frac{u_n(ra(t) + \bar{w}_n(t) + (\rho_n - r)/\gamma_n r)}{r} \tag{4}$$

Time differentiate and use $u_n{}'(c) = -\gamma_n u_n(c)$ to get

$$\dot{V}_n(t) = -\gamma_n V(t)(r\dot{a}(t) + \dot{\bar{w}}_n(t)). \tag{5}$$

During unemployment, $\dot{a}(t) = ra(t) + B(t) - c(t)$. Equations (3) and (4) imply:

$$c(t) = ra(t) + \bar{w}_n(t) + (\rho_n - r)/\gamma_n r, \tag{6}$$

$$\dot{a}(t) = B(t) - \bar{w}_n(t) - (\rho_n - r)/\gamma_n r. \tag{7}$$

Substituting equation (7) into equation (5) yields

$$\dot{V}_n(t) = V_n(t)\Big(-\gamma_n\big(r(B(t) - \bar{w}_n(t)) + \dot{\bar{w}}_n(t)\big) + \rho_n - r\Big). \tag{8}$$

With a reservation wage rule, lifetime utility during unemployment is a function of time and evolves according to

$$\rho_n V_n(t) = u_n(c(t)) + \dot{V}_n(t)$$
$$+ \lambda_n(t)\int_{\bar{w}_n(t)}^{\infty}\left(\frac{u_n(ra(t) + w + (\rho_n - r)/\gamma_n r)}{r} - V_n(t)\right) dF_n(w, t)$$

Use equation (3) to eliminate $u_n(c(t))$, equation (8) to eliminate $\dot{V}_n(t)$, and divide

through by $V_n(t)$ as given in equation (4):

$$\dot{\bar{w}}_n(t) = r(\bar{w}_n(t) - B(t))$$
$$- \frac{\lambda_n(t)}{\gamma_n} \int_{\bar{w}_n(t)}^{\infty} \left(1 - \frac{u_n(ra(t) + w + (\rho_n - r)/\gamma_n r)}{u_n(ra(t) + \bar{w}_n(t) + (\rho_n - r)/\gamma_n r)}\right) dF_n(w, t)$$

Since $u_n(a)/u_n(b) = -u_n(a - b)$, this is equivalent to

$$\dot{\bar{w}}_n(t) = G_n(\bar{w}_n(t), t) - rB(t), \tag{9}$$

where

$$G_n(\bar{w}, t) \equiv r\bar{w} - \frac{\lambda_n(t)}{\gamma_n} \int_{\bar{w}}^{\infty} (1 + u_n(w - \bar{w})) \, dF_n(w, t). \tag{10}$$

This is a crucial equation for our analysis. In a stationary environment, with constant subsidies B, a constant arrival rate λ, and a constant wage distribution $F(w)$, the stationary solution in equation (9) boils down to the reservation wage equations in Shimer and Werning (2005, 2006).

The relations in equations (3)–(9) are necessary conditions for worker optimality. Indeed, they completely characterize behavior over any finite horizon. Together with appropriate "transversality" conditions, which pin down the relevant solution to the ordinary differential equation (9), they are also sufficient for worker optimality. In particular, given a path for benefits $\{B(t)\}$, one solves the ODE (9) for the reservation wage path $\bar{w}(t)$. Then equation (6)–(7) can be solved for the path of assets $a(t)$ and consumption $c(t)$ during unemployment. This then characterizes the entire allocation.

The appropriate solution to equation (9) when primitives $F(w,t)$ and $\lambda(t)$ and benefits $B(t)$ are constant is the constant reservation wage with $\dot{w}(t) = 0$. For the purposes of this paper, we do not need to characterize "transversality" conditions in more general cases for two reasons: (i) since our relations completely characterize behavior over any finite horizon they will suffice for our theoretical results, derived using dynamic programming arguments (for example, Proposition 1 below); (ii) our numerical work truncates, by necessity, the economy by assuming that primitives and benefit policy are constant for $t \geq T$, after some long duration T.

Relation between $\bar{w}_n(t)$ and $B(t)$. We shall use the characterization of the relationship in equation (9) between the reservation wage path $\{\bar{w}_n(t)\}$ and subsidy schedule $\{B(t)\}$ extensively. It is useful to understand what this relation does and does not imply.

Suppose we are in a stationary environment so that $G_n(\bar{w}, t) = G_n(\bar{w})$ is independent of time t. Then, at a steady state, where $\dot{\bar{w}}_n = 0$, we have $B = G_n(\bar{w}_n)/r$. Since G_n is increasing in $\bar{w}_n$ it follows that there is a positive relation between subsidies and the reservation wage. This is intuitive since a higher subsidy makes the option of waiting for higher wage draws more attractive without making employment any more desirable. As a result, the worker becomes more picky about what jobs to accept.

However, the steady-state relationship does not imply that along any dynamic path $\bar{w}_n(t)$ and $B(t)$ will rise and fall in tandem. In equation (9) subsidies $B(t)$ determine both $\bar{w}_n(t)$ and $\dot{\bar{w}}_n(t)$. Informally, if the reservation wage is rising sharply, it indicates that the unemployed worker's lifetime utility is doing the same; things are better in the near future, so current subsidies must be temporarily low.

This implies that there is no simple relation between the paths of $\bar{w}_n(t)$ and $B(t)$. For instance, suppose $\bar{w}_n(t)$ is monotonically increasing; then, it may seem reasonable to expect subsidies $B(t)$ to also be increasing. This will be the case as long as $\bar{w}_n(t)$ does not accelerate too much, so that $\dot{\bar{w}}_n(t)$ does not rise too abruptly; otherwise, equation (19) implies that subsidies will decrease over a range where $\dot{\bar{w}}_n$ rises quickly. Thus, a monotonic $\bar{w}_n(t)$ does not imply monotonic subsidies $B(t)$. The converse, however, is true: if the subsidy schedule $B(t)$ is monotonic then $\bar{w}_n(t)$ is monotonic.

This discussion emphasizes the dynamic nature of worker's search problem. The reservation wage is not only affected by the current subsidy, but also by future subsidies. As a result, current subsidies may generally provide a poor measure of the current subsidy to unemployment implicit in the entire schedule.[4] Perhaps a better measure is simply to observe the effect on the actual reservation wage, which

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[4] To take an extreme example, suppose that subsidies are negative during the first week of unemployment but they then jump up to a positive level, much higher than any potential wage. Few would describe the situation faced by the worker in the first period as providing a tax on unemployment that encourages finding a job.

is forward looking and incorporates the dynamics of future subsidies.

4. Optimal Policy

We imagine an unemployment insurance agency that wishes to maximize a weighted average of unemployed workers' lifetime utility subject to the constraint that it must break even on average. If workers are heterogeneous, unemployment insurance redistributes income across individuals. To focus attention on insurance, we allow for the possibility of lump-sum transfers between groups of workers at time zero. Given the structure of preferences in equation (4), this implies the unemployment insurance agency selects a single subsidy schedule $B(t)$ to maximize the sum of the reservation wages net of the discounted cost the program.

$$\sum_{n=1}^{N} \bar{w}_n(0)\mu_n(0) - rC,$$

where $\bar{w}_n(0)$ is the time-zero reservation wage of type n workers, $\mu_n(0)$ is the measure of type n workers in the unemployed population at time 0, and C is the expected cost of the unemployment subsidy system:

$$C \equiv \int_0^\infty e^{-rt} B(t) \sum_{n=1}^N \mu_n(t) dt.$$

where

$$\mu^n(t) \equiv \exp\left(-\int_0^t H^n(\bar{w}^n(s), s) ds\right) \tag{11}$$

is the probability of being unemployed at time t and

$$H^n(\bar{w}^n, s) \equiv \lambda^n(s)(1 - F^n(\bar{w}^n, s)) \tag{12}$$

is the hazard rate of accepting a job.

The agency recognizes that for any benefit schedule $B(t)$, the reservation wage of type n workers solves equation (9) and the transversality condition, while the

measure of unemployed type n workers decreases as these workers find jobs according to equation (11). The solution to this problem maximizes total surplus, which can then be split, using lump-sum transfers, in any way.

Use equation (9) to eliminate $B(t)$ from the objective function

$$\sum_{n=1}^{N} \bar{w}_n(0)\mu_n(0) - rC = \sum_{n=1}^{N} \bar{w}_n(0)\mu_n(0)$$
$$- \int_0^\infty e^{-rt} \sum_{n=1}^{N} \left(\dot{\bar{w}}_n(t) - G_n(\bar{w}_n(t), t)\right)\mu_n(t)dt \quad (13)$$

Use integration-by-parts to eliminate the term of the integral involving $\dot{\bar{w}}(t)$:

$$\int_0^\infty e^{-rt}\dot{\bar{w}}_n(t)\mu_n(t)dt = -\bar{w}_n(0)\mu_n(0) - \int_0^\infty e^{-rt}\bar{w}_n(t)(\dot{\mu}_n(t) - r\mu_n(t))dt$$
$$= -\bar{w}_n(0)\mu_n(0) + \int_0^\infty e^{-rt}\bar{w}_n(t)\left(r + H_n(\bar{w}_n(t), t)\right)\mu_n(t)dt,$$

where the second line differentiates equation (11) to get

$$\dot{\mu}_n(t) = -\mu_n(t)H_n(\bar{w}_n(t), t). \tag{14}$$

Substitute this back equation (13) to simplify the objective function $\bar{w}(0) - rC$. The planner must choose a sequence of reservation wages for each type to maximize

$$\int_0^\infty e^{-rt} \sum_{n=1}^N J_n(\bar{w}_n(t), t)\mu_n(t)dt, \tag{15}$$

where

$$J_n(\bar{w}_n(t), t) \equiv \bar{w}_n(t)\big(r + H_n(\bar{w}_n(t), t)\big) - G_n(\bar{w}_n(t), t). \tag{16}$$

Since there is a single unemployment subsidy schedule, the evolution of the reservation wages are linked by equation (9):

$$\dot{\Delta}_n(t) = G_n(\bar{w}_1(t) + \Delta_n(t), t) - G_1(\bar{w}_1(t), t), \tag{17}$$

where $\Delta_n(t) \equiv \bar{w}_n(t) - \bar{w}_1(t)$ is the difference between the reservation wage of type n and type 1 workers. Finally, the share of each type evolves according to equation (14).[5]

We can view this as an optimal control problem where the planner chooses a sequence for the reservation wage of type 1 workers and an initial value for other types' reservation wages. This then determines the evolution of the differences $\Delta_n(t)$, $n = 2, \ldots, N$ and the unemployment rates $\mu_n(t)$, $n = 1, 2, \ldots, N$. Of course, the planner also faces a set transversality conditions.

Appendix A discusses our solution method for the case of $N = 2$, when this reduces to an optimal control problem with three state variables and one control variable. We show how to eliminate one of the state variables and apply Pontryagin's Maximum Principle to find the solution.[6]

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[5] Formally, we also require that the the implied worker behavior be optimal. As discussed in Section 3 this requires some "transversality" conditions to be met, which we fortunately do not need to specify for our purposes.

[6] An alternative is to formulate the dynamic programming Hamilton-Jacobi-Bellman partial-differential equation of the problem.

5. Human Capital Depreciation

In this section we consider the case of a single worker type that faces a non-stationary search problem, with λ or F are changing over time. The optimal unemployment insurance problem simplifies considerably:

$$\max_{\{\bar{w}\}} \int_0^\infty e^{-rt} J(\bar{w}(t), t) \mu(t) dt \qquad (18)$$

$$\text{subject to } \dot{\mu}(t) = -\mu(t) H(\bar{w}(t), t),$$

where J is defined in equation (16) and $\mu(0) = 1$ is given.[7]

5.1. Constant and Non-Constant Benefits

An interesting property of optimal subsidies that follows from our reformulation is that the schedule is entirely forward looking: only future values of λ and F are

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[7] Formally, we also require that the implied worker behavior be optimal—which requires "transversality" conditions, as discussed in Section 3.

relevant.[8] The next result then follows immediately from this observation.[9]

Proposition 1 (Shimer-Werning, 2005) *With a single worker type facing a stationary problem $\lambda(t) = \lambda$ and $F(w,t) = F(w)$ for all $t \geq 0$ the optimal subsidy schedule is flat: $B(t) = \bar{B}$ for some $\bar{B} > 0$.*

To tackle the general problem we write it recursively. Let $\Phi(\mu, t)$ be the value function for the problem in equation (18). This value function solves the Bellman equation

$$r\Phi(\mu,t) = \max_{\bar{w}} \Big(J(\bar{w},t)\mu - \Phi_\mu(\mu,t)H(\bar{w},t)\mu + \Phi_t(\mu,t)\Big).$$

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[8] Note that this is true even when we consider explicitly any "transversality" condition required to ensure worker optimality.

[9] This result is proven in Shimer and Werning (2005) in a discrete time version of the model using a different argument. That paper shows that the planner does not want to distort savings. In contrast, here we have simply assumed that the policy problem does not consider introducing such distortions.

Note that the value function is homogeneous of degree one in μ and so we can define $\phi(t) \equiv \Phi(\mu, t)/\mu$ solving

$$r\phi(t) = \max_{\bar{w}} \left(J(\bar{w}, t) - \phi(t)H(\bar{w}, t) + \dot{\phi}(t) \right)$$

Equivalently, we have an ordinary differential equation for $\phi(t)$:

$$\dot{\phi}(t) = M(\phi(t), t) \tag{19}$$

where the law of motion function M is given by

$$M(\phi, t) \equiv \min_{\bar{w}} \left((r + H(\bar{w}, t))\phi - J(\bar{w}, t) \right). \tag{20}$$

Note that the envelope condition implies that the law of motion function $M(\phi, t)$ is increasing in ϕ. Moreover, since the cross-partial derivative of $\bar{w}$ and ϕ is negative in the objective function, it follows that the optimal $\bar{w}$ is increasing in ϕ.

To characterize optimal unemployment insurance, we simply need to solve this ordinary differential equation (19), ensuring that the transversality condition holds. When primitives settle down in the long-run one can solve backwards from the long-run steady-state. The optimal reservation wage solves the right hand side of equation (20) at each date. And the subsidies that implement this reservation wage are found by inverting equation (9) for $B(t)$.

It is instructive to verify how the stationary solution in Proposition 1 solves the ODE system in equation (19). Since primitives are constant the law of motion is independent of time: $M(\psi, t) = M(\psi)$. Since $M(\psi)$ is increasing there exists a unique steady-state value ψ^* satisfying $0 = M(\psi^*)$. Then note that the stationary solution $\psi(t) = \psi^*$ solves the ODE system and satisfies the transversality conditions. The reservation wage and subsidies implicit in this solution are constant, as in Proposition 1.

A simple non-stationary case, illustrated in Figure 1, is when primitives are constant up to some time T, at which point they switch forever after. That is, we have $\lambda(t) = \lambda_0$ and $F(w, t) = F_0(w)$ for all $t < T$ and $\lambda(t) = \lambda_1$ and $F(w, t) = F_1(w)$

for all $t \geq T$. This implies that ϕ evolves according to $\dot{\phi} = M_0(\psi)$ for $t < T$, and then $\dot{\phi} = M_1(\psi)$ for $t \geq T$.

We know that the optimal solution must reach the steady-state point ϕ_1^* of M_1 at $t = T$. Thus, the initial value $\phi(0)$ must start somewhere to the right of point ϕ_0^* and increase—accelerating with the explosive dynamics of M_0—until it reaches ϕ_1^* exactly at time $t = T$, at which point it remains constant there. The larger is T the closer $\phi(0)$ must be to ϕ_0^*; indeed, as $T \to \infty$ then $\phi(0)$ limits to ϕ_0^*.

The implications for subsidies $B(t)$ are immediate translations of those derived for $\phi(t)$. Let B_i^* denote the optimal constant subsidies for the stationary problem with λ_i and $F_i(w)$. Then subsidies $B(t)$ converge to the optimal constant subsidy B_1^* in the long run as $t \to \infty$ and they start somewhere near B_0^*. This result is generalized in the next proposition, where we imagine time extending indefinitely on both sides.

Proposition 2 *Suppose that we have $\lambda(t)$ and $F(w,t)$ defined for all $t \in \mathbb{R}$ with well defined limits $\lim_{t \to -\infty} \lambda(t) = \lambda_0$ and $\lim_{t \to -\infty} F(w,t) = F_0(w)$ and $\lim_{t \to \infty} \lambda(t) = \lambda_1$ and $\lim_{t \to \infty} F(w,t) = F_1(w)$. Then $B(t)$ is such that $\lim_{t \to -\infty} B(t) = B_0^*$ and*

$\lim_{t\to\infty} B(t) = B_1^*$, *where B_i^* is defined as the optimal constant subsidy levels for the economies with constant primitives at λ_i and $F_i(w)$.*

medskip**An Aside: Q-Theory Analogy.** Our model can be mapped into the adjustment cost model of investment with constant returns to scale which Hayashi (1982) used to related investment to "Tobin's Q".

In the case of certainty the investment model can be formulated as maximizing discounted profits

$$\int_0^\infty \pi(i(t), t) K(t) dt$$

subject to $\dot{K}(t) = i(t)K(t)$, where $i(t) = I(t)/K(t)$ is the investment rate. There are constant returns to scale in the net profit function, which equals revenues net of investment costs, and constant returns in investment. No assumption of concavity is required. One can show that the value function is homogeneous of degree one, $V(K) = qK$, so that the marginal and average value of capital, q, often referred to as "Tobin's Q", solves

$$rq = \max_i \{\pi(i, t) + iq\} + \dot{q}.$$

The important result for this theory is that the investment rate is a function of q; the entire future is captured by this forward looking variable.

This model maps directly into our framework, with μ playing the role of K, ϕ playing the role of q, $\bar{w}$ playing the role of investment i, and π given by $\bar{w}(r + H(\bar{w}, t)) - G(\bar{w}, t)$.

5.2. Numerical Explorations

This section describes the outcome of two numerical experiments. We first consider skill depreciation, then search depreciation. The purpose of these explorations is not to obtain definite quantitative conclusions. Our goal is to understand the qualitative workings of the model and perhaps get a tentative feel for their quantitative importance.

Our baseline parameterization is close to that in Shimer and Werning (2005). We set $\gamma = 1$, $r = \rho = .001$ and $\lambda = 1$ and interpret a period to be a week, with an implied annual interest rate of 5.3%. The distribution is assumed to be Fréchet:

$F(w) = \exp(-w^{-\theta})$. We set $\theta = 103.5$.

This baseline calibration has wages concentrated near 1 and delivers an expected duration of around 10 weeks, which is in line with unemployment durations in the United States. The optimal constant subsidy in this economy turns out to be very low, around 0.01. The desire to insure is small enough, while the moral-hazard problem severe enough, that low subsidies result at the optimum. As discussed by Shimer and Werning (2005), liquidity, in contrast, is important: workers are able to smooth their shocks, spreading their impact over time, by dissaving or borrowing.

- **Skill Depreciation**

In this first exercise we keep $\lambda(t) = 1$ constant and instead assume that the distribution of wages shifts downwards in a parallel fashion. Our specification is inspired by the depreciation process used in Ljungqvist and Sargent (1998). Specifically we

let

$$F(w,t) = F(w - \exp(-\delta_F \cdot t)) \qquad t < T$$

$$F(w,t) = F(w - \exp(-\delta_F \cdot T)) \qquad t \geq T$$

where $F(w)$ is the baseline Fréchet distribution defined above. Thus, at $t = 0$ the wage distribution is simply a rightward shift of the baseline distribution. Over time the distribution shifts continuously to the left, converging back to the baseline distribution. We set the speed of convergence to $\delta_F = 0.01$.

Our approach is to solve the ODE system in equation (19). We first solve the system's steady state for $t \geq T$. We then solve the ODE backwards up to $t = 0$. This gives us $\phi(t)$. We then compute $\bar{w}(t)$ and solve equation (9) for $B(t) = \frac{1}{r}(G(\bar{w}(t), t) - \dot{\bar{w}}(t))$.[10]

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[10] The numerical details are as followed. We employed Matlab's `ode45` routine to solve the ODE, along with the `fminbnd` routine for the required optimization. We computed $\phi(t)$ and then backed out the implied reservation wage $\bar{w}(t)$. We then fit a piecewise cubic shape-preserving spline through $\bar{w}(t)$ (using Matlab's `interp1` routine with the `pchip` option) to obtain the derivative $\dot{\bar{w}}(t)$

Figure 2 shows the outcome of this exercise for the optimal schedule of subsidies. The schedule is decreasing with unemployment duration, starting at subsidies just above 0.30 and falling to the steady-state value of 0.01—equal to the value of subsidies at the baseline. Figure 4 shows that these subsidies induce the reservation wage to fall during the unemployment spell. The rate at which the reservation wage drops, however, does not keep up with the rate of decline in the distribution of wages. This is shown in Figure 5 where we plot the acceptance probability of job opportunities, $1 - F(\bar{w}(t))$. The optimal schedule induces workers to become pickier: the probability of accepting a job offer falls from around 80% to 10%. Note that in this case, since $\lambda(t) = 1$ the acceptance probability equals the hazard rate of out unemployment, $\lambda(t)(1 - F(\bar{w}(t)))$. Benefits, however, all even faster: Figure 3 plots the ratio $B(t)/\bar{w}(t)$, it is decreasing.

As Figure 5 illustrates the hazard rate out of unemployment is high, especially at the beginning of the spell where depreciation is greatest. Workers become more willing to accept bad matches to prevent their skills from depreciating. We com-

from it.

puted the solution for $\theta = 15$ which increases the dispersion of wages making search more attractive. Figure 6 plots optimal subsidies for this case. Once again we see that subsidies are decreasing. Figure 7 shows that the hazard rate out of unemployment is much lower now, and close to the hazard at the benchmark without depreciation.

• Search Depreciation

In this second exercise we assume that the wage distribution is fixed at the baseline's Fréchet specification, but that the arrival rate falls continuously over time. Specifically, we let $\lambda(t) = \bar{\lambda}_0 \exp(-\delta_\lambda \cdot t) + \bar{\lambda}_1$ for $t \leq T$ and constant thereafter; we set $\delta_\lambda = 0.01$, $\bar{\lambda}_0 = .9$ and $\bar{\lambda}_1 = .1$, so that the arrival rate starts at our benchmark of one offer a week and falls continuously towards one offer every 10 weeks.

Figure 8 shows the results of this exercise for optimal subsidies. We find an increasing schedule, which is in line with Proposition 2, since a lower arrival rate increases the duration risk of unemployment prompting higher subsidies. The level of subsidies is quite low in this case throughout, so the increase in subsidies is

not very spectacular. Figure 9 shows that even though subsidies rise the acceptance probability (blue line) rises with duration as job opportunities become rarer. Workers become less picky and lower their reservation wages. However, the resulting hazard rate out of unemployment $\lambda(t)(1 - F(\bar{w}(t)))$, also shown (green line), comes out to be slightly declining.

We have found that the limiting long-run subsidy is quite sensitive to the long-run value λ_1 for low enough values, and become quite large for λ_1 near zero. To illustrate this Figure 10 shows optimal subsidies when $\lambda_1 = .01$. Note that subsidies rise only moderately for about 6 years—similar to what we found in Figure 8 for higher λ_1. However, when $\lambda(t)$ gets very close to zero, subsidies rise sharply and asymptote to a high level, around 0.73. Of course, with these parameters only an insignificant fraction of workers make it this far into long-term unemployment. Nevertheless, this illustrates that the magnitude of the increase in subsidies depends on parameters of the problem.

As explained in Section 3, when the reservation wage accelerates there may be regions where subsidies are decreasing. We found that this occurs over an interme-

diate region of time for the extreme parameter value of $\lambda_1 = 0$. Figure 12 shows that subsidies are decreasing over an intermediate region. Indeed, they become slightly negative there because they were low and close to zero initially. The non-monotonicity occurs after around 6 years, a duration that an insignificant fraction of workers will reach. However, this illustrates the point we made earlier: that subsidies may be a poor measure of the subsidy to unemployment since workers are forward looking and incorporates the dynamics of future subsidies.

6. Heterogeneity

When workers are heterogeneous, for any given common subsidy schedule, the pool of unemployed worker types typically varies over time. That is, worker types that tend to have lower hazard rates become more prevalent as time passes. This section focuses on how this affects the optimal subsidy schedule. We study the case with two types of workers indexed by $n = 1, 2$. To bring out the role of heterogeneity we suppose that for each worker type the arrival rate of offers λ_n and the distribution

of wages F_n does not vary with time t.

6.1. Numerical Explorations

As with the depreciation case, the purpose of our explorations is not to obtain definite quantitative conclusions but to understand the qualitative workings of the model.

In all our exercises we start with equal population fractions for both worker types $\mu_1 = \mu_2 = 1/2$. We begin with heterogeneity in the distribution functions of potential wages. All the other parameters are set at our baseline, as described in Section 5. If we could costlessly separate workers, then it would be optimal to give worker n some constant subsidies equal to $\bar{B}_n$. The values of $\bar{B}_n$ will be useful references in the exercises below.

Heterogeneity in θ. In this experiment we set $\theta_1 = 50$ and $\theta_2 = 100$, in our Fréchet specification: $F_n(w) = \exp(-w^{-\theta_n})$. Workers of type 1 tend to have a lower hazard rate out of unemployment because they sample from a distribution with

thicker tails and have a higher option value of waiting for offers. As a consequence, their fractions rises in the unemployment pool over time. The implied constant optimal subsidies are small for both workers, but slightly higher for type-1 workers: $\bar{B}_1 = 0.024$ and $\bar{B}_2 = 0.012$.

Figure 14 shows the optimal subsidy schedule for this case. Overall subsidies start around $\bar{B}_2$ rise with duration and limit to $\bar{B}_1$, as type-1 workers prevail in the unemployment pool. Indeed, subsidies rise and initially overshoot $\bar{B}_1$ and then come down back towards it. Figure 15 shows the evolution of the fraction of type 1 workers $\mu_1(t)/(\mu_1(t) + \mu_2(t))$ alongside the relative hazard rate $H_1(t)/H_2(t)$. In this example the relative hazard rate is quite constant, hoovering around $1/2$, and the fraction of type-1 workers rises steadily towards 1; the hazard rates are also quite constant and around $H_1 \approx .48$ and $H_2 \approx .95$.

In this example, given the chosen parameters, the overall subsidy levels are quite low. As a consequence, so is the rise in subsidies. This need not be the case: Figure 16 and 17 are comparable to Figure 14 and 15 but using $\theta = 10$ and $\theta = 20$. The associated constant subsidy levels are now $\bar{B}_1 = 0.267$ and $\bar{B}_2 = 0.073$.

Figure 16 shows that the rise in subsidies is more significant now as subsidies rise from a value near $\bar{B}_1$ and asymptote to $\bar{B}_2$.

In the previous two examples workers with lower hazard rates were also those workers that merited a higher constant unemployment subsidy, in the sense that $\bar{B}_1 > \bar{B}_2$. As a consequence, when policy cannot separate workers subsidies tend to rise over time. Our next example inverts this logic.

Parallel Shifts. We now assume that workers of type 1 sample from a distribution that is shifted to the left in a parallel fashion, so that $F_1(w) = F_2(w + \omega)$.

One interpretation for the source of heterogeneity is that the wage workers sampled is to be interpreted broadly as a wage net of work effort cost on the job. Then workers of type 1 can be interpreted as "lazy" relative to type 2, in that they have a higher cost of working.

Another interpretation is that type 1 workers have suffered a discrete loss in human capital, lowering their productivity on the job by a constant amount. This is similar to the model in Ljungqvist and Sargent (1998), where a fraction of workers suffer an immediate loss in skills upon being laid off, the rest are spared. This

introduces a form of heterogeneity that is very similar to that explored here. Their model also includes a steady rate of depreciation during the unemployment spell, that can be roughly captured by our previous analysis.[11] Thus, we can captures both elements, heterogeneity and depreciation, in Ljungqvist and Sargent's (1998) specification.

We solve the model with F_2 set to the Fréchet specification with $\theta = 10$ and $\omega = 1/2$. These choices imply $\bar{B}_1 = 0.185$ and $\bar{B}_2 = 0.242$. The results are shown in Figure 18 and Figure 19. Benefits fall from around 0.225, not too far from $\bar{B}_2$, and asymptote to the lower value of $\bar{B}_1$.

In all these examples subsidies start somewhere between $\bar{B}_2$ and $\bar{B}_1$ and eventually converge to $\bar{B}_1$. This is intuitive since workers of type 1 prevail in the unemployment pool in the long-run. Note, however, that in all these examples subsidies actually overshoot the B_2. Note that such overshooting never occurred in our explorations with homogeneous workers suffering from human capital depreciation.

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[11] One difference is that they model the steady depreciation during the spell as stochastic instead of deterministic, partly for numerical reasons.

Our intuition for this finding is as follows. The planner would wish to separate workers by types and offer them different unemployment subsidy levels, i.e. $\bar{B}_1$ and $\bar{B}_2$. Although he cannot do so there is something that imperfectly mimics separation. To see this, note that when a worker rejects a wage to continue searching and collecting subsidies, the relevant per-period subsidy in her calculation is an average of the unemployment subsidies over the span of time until the next suitable job is found. Since workers of type 1 have lower hazard rates out of unemployment, this subsidy reflects a longer average of subsidies.

It follows that, all things the same, a downward sloping subsidy schedule implies a lower effective subsidy for type 1 workers than for type 2 workers; the reverse is true of an increasing schedule. Thus, titling the schedule helps imitate desired discriminatory policy. It is this titling that is responsible for the overshooting: subsidies rise or fall past the desired ex-post long-run level in order to provide better incentives ex ante.

Note that, by the same logic, this phenomena is symptomatic of a time-consistency issue: ex post the overshot subsidies are not optimal, they were provided to af-

fect incentives ex ante. Formally, as we discussed briefly above (and in detail in Appendix A) the planning problem can be formulated with a two-dimensional state vector using the fraction of type 1 workers $\mu_1(t)/(\mu_1(t) + \mu_2(t))$ and the promised difference in reservation wages $\bar{w}_2(t) - \bar{w}_1(t)$. However, only the first is a physically unalterable state variable, the second is, what is sometimes termed, a 'pseudo' state variable, that is is introduced to render the commitment planning problem recursive (i.e. to keep track of all "promise-keeping" constraints). In general, the promised differences in reservation wages will be inefficient ex post: the planner would wish to reoptimize over $\bar{w}_2(t) - \bar{w}_1(t)$, ignoring the inherited promised value.

Recall that, in contrast, when workers are homogeneous the planning problem can be reduced so that at any point in time it is entirely forward-looking. There are no state variables except for calendar time (and none whatsoever when the problem is stationary). It does not, therefore, feature a time inconsistency issue.

7. Conclusions

This paper provided a tractable framework for studying the optimal timing of unemployment subsidies over the unemployment spell. Subsidies should be constant when workers are homogeneous and face a stationary search problem. In contrast, human capital depreciation and worker heterogeneity can lead to increasing or decreasing schedules, depending on the precise nature of the depreciation or heterogeneity; we provided some useful dichotomies. A simple heuristic is suggested by our finding that optimal subsidies generally shift in the direction suggested by a comparative-static analysis.

Appendix

A. Optimal Control with Heterogeneous Agents

In this appendix we describe in detail the optimal control problem with two worker types, its associated Hamiltonian, the resulting optimality conditions, and our numerical strategy for solving it. We being with some definitions.

Definitions. Let the difference in reservation wages be $\Delta(t) \equiv \bar{w}_2(t) - \bar{w}_1(t)$. Then the law of motion for this difference is

$$\dot{\Delta}(t) = G_2(\bar{w}_1 + \Delta) - G_1(\bar{w}_1), \tag{21}$$

Define the fraction of type 1 workers as

$$\alpha(t) \equiv \frac{\mu_1(t)}{\mu_1(t) + \mu_2(t)},$$

with the law of motion

$$\dot{\alpha}(t) = \alpha(t)\big(1 - \alpha(t)\big)\Big(H_2\big(\bar{w}_1(t) + \Delta(t)\big) - H_1\big(\bar{w}_1(t)\big)\Big). \tag{22}$$

The objective function is then

$$\int_0^\infty e^{-rt}\Big(\alpha(t)J_1\big(\bar{w}_1(t)\big) + \big(1 - \alpha(t)\big)J_2\big(\bar{w}_1(t) + \Delta(t)\big)\Big)\mu(t)dt, \tag{23}$$

where $J_n(\bar{w}) \equiv \bar{w}_n\big(r + H_n(\bar{w}_n)\big) - G_n(\bar{w}_n)$. Thus, the planning problem is to maximizing equation (23) over $\bar{w}_1(t)$, $\Delta(t)$, $\alpha(t)$ and $\mu(t)$ subject to equation (21), equation (22) and

$$\dot{\mu}(t) = -\mu(t)(\alpha(t)H_1(\bar{w}_1(t)) + (1 - \alpha(t))H_2(\bar{w}_1(t) + \Delta(t))).$$

It is useful to think of $\bar{w}_1(t)$ as the control and $\Delta(t)$, $\alpha(t)$ and $\mu(t)$ as state variables.

Truncated Control Version. In practice, we solve a version of the problem where

policy is restricted to offering constant subsidies for $t \geq T$, for T large. Let $\Psi(\alpha, \Delta)$ be the continuation value at T with a constant subsidy that implements a difference in reservation wages of Δ and with the fraction of type 1 workers equal to α. The problem can be written as

$$\int_0^T e^{-rt} \bar{J}(\alpha(t), \bar{w}_1(t), \Delta(t), t)\mu(t)dt + e^{-rT}\Psi(\alpha(T), \Delta(T), T)\mu(T). \quad (24)$$

with $\alpha(0) = \alpha_0$ and $\mu(0) = \mu_0$ are given, where

$$\bar{J}(\alpha, \bar{w}_1, \Delta, t) \equiv \alpha J_1(\bar{w}_1, t) + (1 - \alpha)J_2(\bar{w}_1 + \Delta, t)$$

Hamiltonian. The Hamiltonian for this problem is (omitting the arguments in functions to save on notation)

$$\mathcal{H} \equiv \bar{J}\mu + \nu_\mu M^\mu \mu + \nu_\alpha M^\alpha + \nu_\Delta M^\Delta$$

where

$$M^{\mu}(\alpha, \bar{w}_1, \Delta, t) \equiv -(\alpha H_1(\bar{w}_1) + (1-\alpha)H_2(\bar{w}_1 + \Delta))$$

$$M^{\alpha}(\alpha, \bar{w}_1, \Delta, t) \equiv \alpha(1-\alpha)\big(H_2(\bar{w}_1 + \Delta) - H_1(\bar{w}_1)\big)$$

$$M^{\Delta}(\bar{w}_1, \Delta, t) \equiv G_2(\bar{w}_1 + \Delta, t) - G_1(\bar{w}_1, t)$$

The functions M^{α} and M^{Δ} are the laws of motion for α and Δ derived above. Note that $\mu \cdot M^{\mu}$ is the law of motion for μ and that M^{μ} itself can be thought as a "normalized" law of motion.

Maximum Principle. The reservation wage solves

$$\max_{\bar{w}_1} \mathcal{H} \tag{25}$$

And the co-states evolve according to

$$\dot{\nu}_\mu = r\nu_\mu - (\bar{J} + \nu_\mu M^\mu) \tag{26}$$

$$\dot{\nu}_\Delta = r\nu_\Delta\mu - \mathcal{H}_\Delta \tag{27}$$

$$\dot{\nu}_\alpha = r\nu_\alpha\mu - \mathcal{H}_\alpha \tag{28}$$

Equations (25)–(28), together with the law of motion equations for μ, α and Δ, comprise an ODE system for the 6-dimensional vector $(\mu, \alpha, \Delta, \nu_\mu, \nu_\alpha, \nu_\Delta)$.

Normalized System. It proves convenient to work with the *normalized* Hamiltonian and co-states:

$$\tilde{\mathcal{H}} \equiv \bar{J} + k_\mu M^\mu + k_\alpha M^\alpha + k_\Delta M^\Delta$$

$$k_\mu \equiv \nu_\mu \qquad\qquad k_\alpha \equiv \frac{\nu_\alpha}{\mu} \qquad\qquad k_\Delta \equiv \frac{\nu_\Delta}{\mu}$$

The benefit of this transformation is that we can reduce the 6-dimensional problem to a 5-dimensional one by dropping μ.

Since μ can be factored out, the optimality condition (25) is equivalent to

$$\max \tilde{\mathcal{H}} \tag{29}$$

and that using (26)–(28) we can obtain the laws of motion for the transformed co-states becomes

$$\dot{k}_\mu = rk_\mu - (\bar{J} + k_\mu M^\mu) \tag{30}$$

$$\dot{k}_\Delta = k_\Delta(r - M^\mu) - \tilde{\mathcal{H}}_\Delta \tag{31}$$

$$\dot{k}_\alpha = k_\alpha(r - M^\mu) - \tilde{\mathcal{H}}_\alpha \tag{32}$$

Equations (29)–(32), together with the law of motion equations for α and Δ, comprise an ODE system for the 5-dimensional vector $(\alpha, \Delta, \nu_\mu, \nu_\alpha, \nu_\Delta)$.

Terminal Conditions. Since $\Delta(0)$ is a free variable its costate should be initially zero

$$k_\Delta(0) = 0 \tag{33}$$

A $t = T$ we must meet the terminal conditions that

$$k_\mu(T) = \Psi(\alpha(T), \Delta(T), T) \tag{34}$$

$$k_\Delta(T) = \Psi_\Delta(\alpha(T), \Delta(T), T) \tag{35}$$

$$k_\alpha(T) = \Psi_\alpha(\alpha(T), \Delta(T), T) \tag{36}$$

Algorithm. The idea is to start at T and work backwards. The control variable is $\bar{w}_1$ and the states are μ, Δ and α; along with their co-states k_μ, k_Δ and k_α. As before, μ plays no role in the dynamics and it can be ignored.

One way to solve this is to guess the values of $\alpha(T)$ and $\Delta(T)$. One can then obtain the values of the co-states at T using equations (34)–(36). This is then sufficient to solve the system backwards for $\bar{w}_1$, α, Δ, k_μ, k_Δ, k_α. One can then compute the implied value of $k_\Delta(0)$ and $\alpha(0)$. One can then search for the initial guess $\alpha(T)$ and $\Delta(T)$ that has $k_\Delta(0) = 0$ and $\alpha(0) = \alpha_0$, that is to satisfy the optimality condition equation (33) and the given initial condition for α.

Useful Expressions. We now collect some expressions that are needed. First,

some derivatives of the Hamiltonian function:

$$\mathcal{H}_\alpha = J_1 - J_2 - (H_1 - H_2)\left((1 - 2\alpha)k_\alpha + k_\mu\right)$$

$$\mathcal{H}_\Delta = (1 - \alpha)\left(J_2' - k_\mu H_2'\right) + k_\Delta G_2' + k_\alpha \alpha(1 - \alpha)H_2'$$

We also need the derivatives:

$$J'(\bar{w}) = r + H(\bar{w}) + \bar{w}H'(\bar{w}) - G'(\bar{w})$$

$$G'(\bar{w}) = r - \lambda(t)\int_{\bar{w}}^{\infty} u_n(w - \bar{w})dF(w, t) = r + \gamma(G(\bar{w}) - r\bar{w}) + \lambda(1 - F(\bar{w}))$$

We define the value from a steady state policy as

$$\Psi(\alpha, \Delta) \equiv \max_{\bar{w}_1}\left(\alpha S_1(\bar{w}_1) + (1 - \alpha)S_2(\bar{w}_1 + \Delta)\right)$$

subject to, $0 = M^\Delta(\bar{w}_1, \Delta)$, where we define for convenience the steady state surplus

function:

$$S_n(\bar{w}) \equiv \bar{w} - \frac{G_n(\bar{w})}{r + H_n(\bar{w})}.$$

It follows immediately that $\Psi(\cdot, \Delta)$ is always differentiable and

$$\Psi_\alpha(\alpha, \Delta) = S_1(\bar{w}_1) - S_2(\bar{w}_1 + \Delta)$$

As for the other partial derivative, note that the constraint $0 = M^\Delta(\bar{w}_1, \Delta)$ can be solved for Δ as a function of $\bar{w}_1$, which we write as $\Delta^*(\bar{w}_1)$. If this function is invertible then the maximization is trivial. In general this function may be non-monotone, so that the above maximization is nontrivial. This may also introduce kinks in the value function $\Psi(\alpha, \cdot)$. However, at points of differentiability we must

have

$$\Psi_\Delta(\alpha, \Delta) = (\alpha S_1'(\bar{w}_1) + (1-\alpha)S_2'(\bar{w}_1 + \Delta)) \frac{1}{\Delta'(\bar{w}_1)} + (1-\alpha)S_2'(\bar{w}_1 + \Delta)$$

$$= (\alpha S_1'(\bar{w}_1) + (1-\alpha)S_2'(\bar{w}_1 + \Delta)) \frac{-M_\Delta^\Delta}{M_{\bar{w}_1}^\Delta} + (1-\alpha)S_2'(\bar{w}_1 + \Delta).$$

References

Hayashi, Fumio, "Tobin's Marginal q and Average q: A Neoclassical Interpretation," *Econometrica*, 1982, *50* (1), 213–24. 32

Hopenhayn, Hugo and Juan Pablo Nicolini, "Optimal Unemployment Insurance," *Journal of Political Economy*, 1997, *105* (2), 412–438. 2, 10

Ljungqvist, Lars and Thomas Sargent, "The European Unemployment Dilemma," *Journal of Political Economy*, June 1998, *106* (3), 514–550. 5, 9, 34, 42, 43

McCall, John, "Economics of Information and Job Search," *Quarterly Journal of Economics*, 1970, *84* (1), 113–126. 11

Pavoni, Nicola, "Optimal Unemployment Insurance, with Human Capital Depreciation and Duration Dependence," 2003. 10

Shavell, Steven and Laurence Weiss, "Optimal Payment of Unemployment-Insurance Benefits over Time," *Journal of Political Economy*, 1979, *87* (6), 1347–1362. 2, 10

Shimer, Robert and Iván Werning, "Liquidity and Insurance for the Unemployed," Staff Report 366, Federal Reserve Bank of Minneapolis December 2005. 2, 3, 10, 11, 12, 18, 28, 33, 34

_ **and Iván Werning**, "Reservation Wages and Unemployment Insurance," 2006. Mimeo, MIT. 11, 18

Violante, Giovanni L. and Nicola Pavoni, "Optimal Welfare-to-Work Programs," 2005. 10

Werning, Iván, "Optimal Unemployment Insurance with Unobservable Savings," 2002. Mimeo. 10
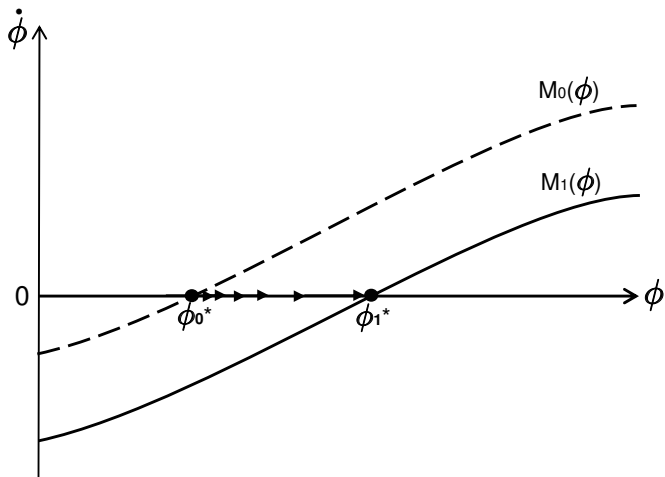
Figure 1: Law of Motion for ϕ.
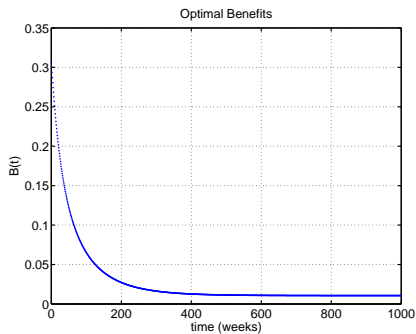
Figure 2: Optimal Benefits with Skill Depreciation.
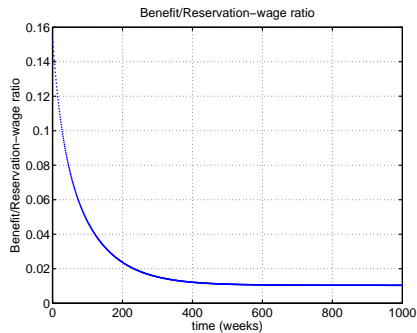
Figure 3: Ratio of subsidies to the reservation wage: $B(t)/\bar{w}(t)$.
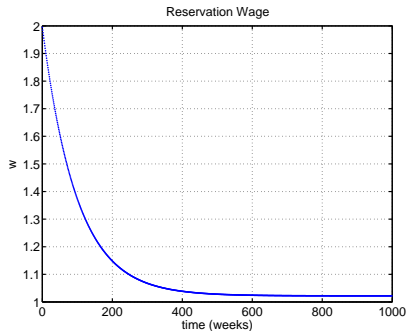
Figure 4: Reservation Wage.
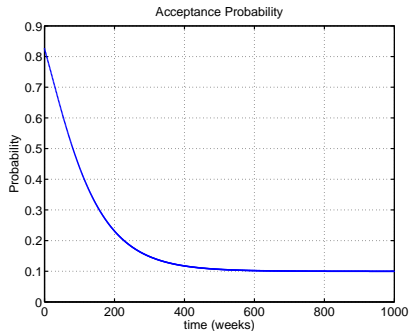
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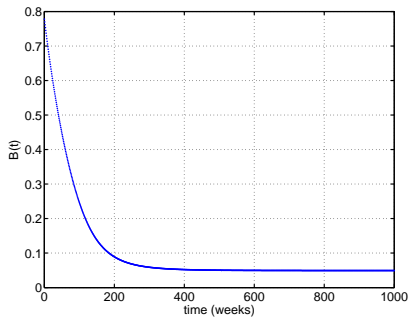
Figure 6: Optimal Benefits with $\theta = 15$.
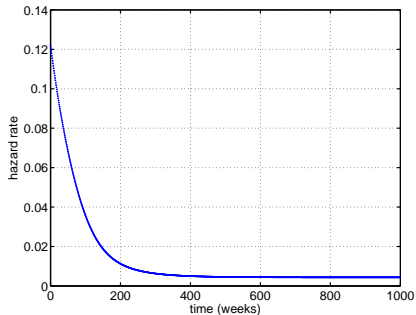
Figure 7: Hazard rate.
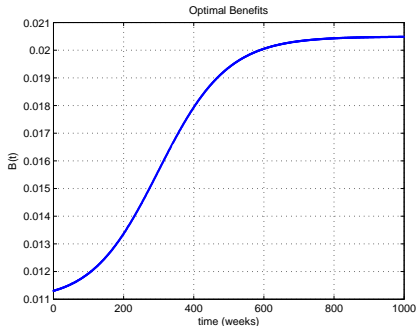
Figure 8: Optimal Benefits with Search Depreciation.
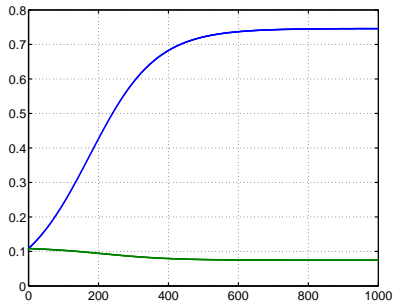
Figure 9: Acceptance Probability and Hazard Rate with Search Depreciation.
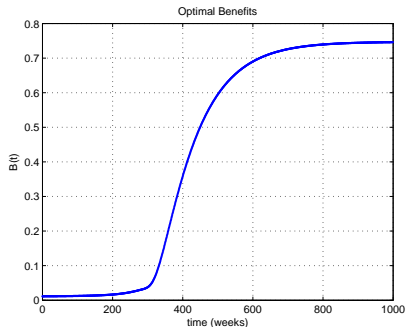
Figure 10: Optimal Benefits with Search Depreciation when $\lambda_1 = .01$.
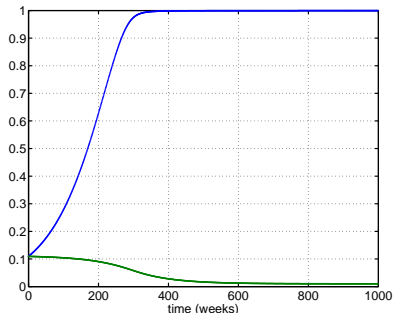
Figure 11: Acceptance Probability and Hazard Rates when $\lambda_1 = .01$.
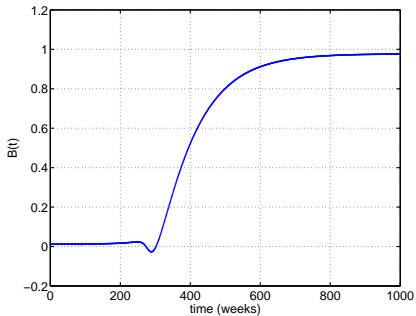
Figure 12: Optimal Benefits with Search Depreciation when $\lambda_1 = .01$.
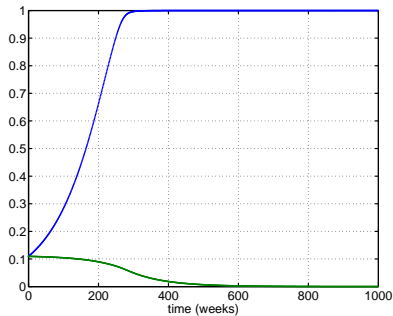
Figure 13: Acceptance Probability and Hazard Rates when $\lambda_1 = .01$.
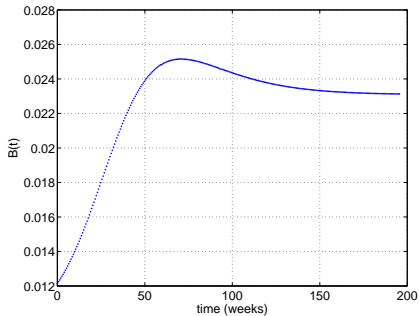
Figure 14: Optimal Benefits with Heterogeneity in θ; $\theta_1 = 50$, $\theta_1 = 100$.
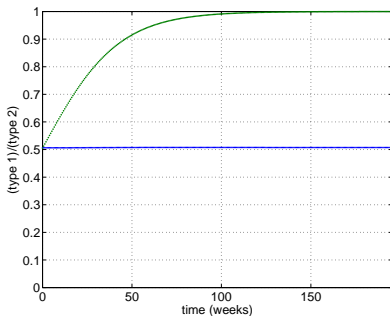
Figure 15: Fraction of type-1 workers and relative hazard rates.
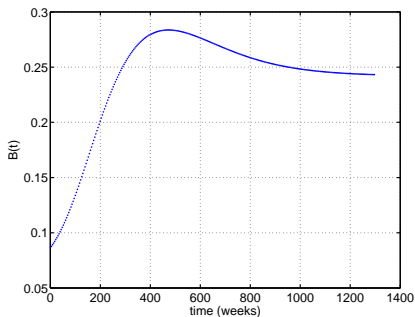
Figure 16: Optimal Benefits with Heterogeneity in θ; $\theta_1 = 50$, $\theta_1 = 100$.
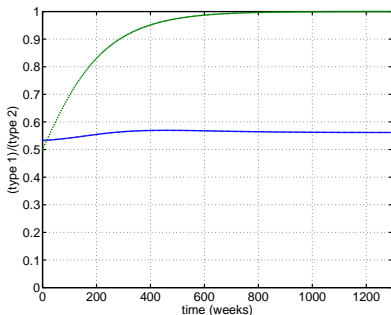
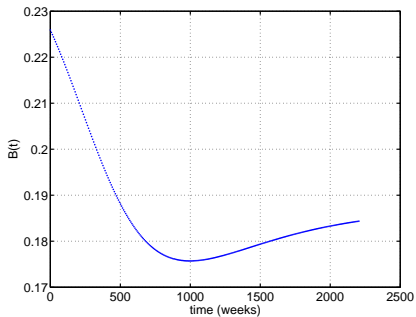Figure 17: Fraction of type-1 workers and relative hazard rates.

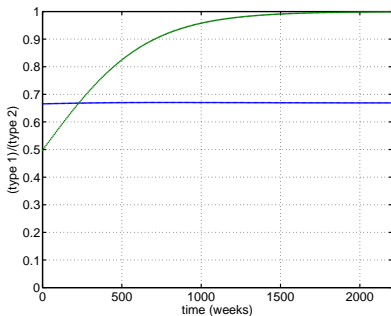Figure 18: Optimal Benefits with Heterogeneity in support of $F(w)$.

Figure 19: Fraction of type-1 workers and relative hazard rates.