PROBABILITY MODELS FOR ECONOMIC DECISIONS by Roger Myerson

excerpts from Chapter 8: Risk Sharing and Finance

## 8.1. Optimal risk sharing in a partnership of individuals with constant risk tolerance

To introduce the basic ideas of optimal risk sharing, let us begin with an example of two individuals (numbered 1 and 2) who are considering a real-estate development project. Suppose that they have an option to buy a tract of land for $125,000, after which they would then need to spend an additional $40,000 on improvements (including an allocation for the cost of their own time in supervising the project) before they could sell the land in subdivided lots. The total revenue that they could then earn from selling these lots would be uncertain, but has an expected value of $200,000 and a standard deviation of $25,000. For simplicity, let us assume here that the time to complete this real estate project is small enough that we can ignore the interest costs of borrowing money to cover the expenses before the revenues come in. So the net returns from this real estate project next year will have expected value

$$\mu = 200{,}000 - (125{,}000 + 40{,}000) = \$35{,}000$$

and standard deviation

$$\sigma = \$25{,}000.$$

Suppose that each of these two individuals evaluates risky incomes using a utility function with constant risk tolerance, where individual 1 has risk tolerance

$$r_1 = \$20{,}000,$$

and individual 2 has risk tolerance

$$r_2 = \$30{,}000.$$

They must decide whether to undertake this real estate project, and if so, how to divide the returns among themselves. Let us assume that the uncertainty about profits from this project can be described by a Normal distribution.

When an individual with constant risk tolerance r has a gamble that will pay a random amount of money drawn from a Normal probability distribution with mean μ and standard deviation σ, his certainty equivalent for the gamble is

$$CE = \mu - (0.5/r)*\sigma\verb|^|2$$

So if individual 1 were to undertake this project himself, his certainty equivalent would be

$$\mu - (0.5/r_1)*(\sigma\text{^}2) = 35000 - (0.5/20000)*(25000\text{^}2)$$

$$= 35000 - 15625 = \$19{,}375.$$

That is, the option to buy this land and undertake this project would be worth $19,375 to individual 1, if he had to undertake all the risks of the project alone.

If individual 2 were to undertake this project by herself, then its value to her would be

$$\mu - (0.5/r_2)*(\sigma\text{^}2) = 25000 - (0.5/30000)*(25000\text{^}2)$$

$$= 35000 - 10417 = \$24{,}583$$

So if individual 1 had the option to buy this land, then individual 2 would be willing to pay up to $24,583 to buy the option from him, and individual 1 would be glad to sell the option for any price above $19,375. Of course it is not surprising that this risky project should be more valuable to the individual who has greater risk tolerance.

But even though individual 2 is strictly more risk tolerant than individual 1, the project could be even more valuable to these individuals if they undertake the project as partners, with individual 1 taking a positive share of the project's risks. For example, if they each took 50% of the net profits from the project, then each individual would anticipate a payment drawn from a Normal distribution with mean 0.50*35000 = $17,500 and standard deviation 0.50*25000 = $12,500. For his 50% share, individual 1 would have certainty equivalent

$$CE(1) = 17500 - (0.5/20000)*(12500\text{^}2) = 17500 - 3906 = \$13{,}594,$$

For her 50% share, individual 2 would have certainty equivalent

$$CE(2) = 17500 - (0.5/30000)*(12500\text{^}2) = 17500 - 2604 = \$14{,}896.$$

So the total certainty-equivalent value of the project to the two individuals when they share it equally is

$$CE(1) + CE(2) = 13594 + 14896 = \$28{,}490$$

Thus, the project is worth more to them when it is shared equally than when the more risk tolerant individual 2 owns it completely.

Such risk sharing is beneficial because each individual j's risk premium $(0.5/r_j)*\sigma\text{^}2$ is proportional to the square of the standard deviation (the variance) of his or her income. So halving individual 2's share from 100% to 50% would halve the standard deviation of her monetary returns from $25,000 to $12,500, which in turn would reduce her risk premium to a quarter of its former value from $10,417 to $2604. This decrease in individual 2's risk premium from giving up 50% of the project (10417−2604 = 7813) is much greater than the increase in individual 1's risk premium when he takes on 50% of the project (3906−0).

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | Suppose profits will be drawn from a Normal distribution | | | | | | | |
| 2 | Mean | 35000 | | | | | | |
| 3 | Stdev | 25000 | | | | | | |
| 4 | | | | | | | | |
| 5 | Profits can be shared by individuals 1 and 2 | | | | | | | |
| 6 | Individ | RiskTol | %Share | Mean | Stdev | CE | RiskPremium | |
| 7 | 1 | 20000 | 0.4 | 14000 | 10000 | 11500 | 2500 | |
| 8 | 2 | 30000 | 0.6 | 21000 | 15000 | 17250 | 3750 | |
| 9 | | | | | | | | |
| 10 | | Sum(RTs) | CE(total,sumRTs) | | | Sum(CEs) | Sum(RPs) | |
| 11 | | 50000 | 28750 | | | 28750 | 6250 | |
| 12 | SOLVER: Maximize F11 by changing C7. | | | | | | | |
| 13 | FORMULAS | | | | | | | |
| 14 | C8. =1-C7 | | | | F11. =SUM(F7:F8) | | | |
| 15 | D7. =C7*$B$2 | | | | G11. =SUM(G7:G8) | | | |
| 16 | E7. =C7*$B$3 | | | | B11. =SUM(B7:B8) | | | |
| 17 | F7. =D7-(0.5/B7)*E7^2 | | | | C11. =B2-(0.5/B11)*B3^2 | | | |
| 18 | G7. =D7-F7 | | | | | | | |
| 19 | D7:G7 copied to D8:G8 | | | | | | | |

**Figure 8.1. Sharing a Normal gamble.**

The spreadsheet in Figure 8.1 is set up to analyze the effect on the individuals' certainty of other ways of sharing the risks of this project. When we enter individual 1's share of the risks into cell C7, then the expected value and standard deviation of 1's income are calclulated in cells D7 and E7 by the formulas `=C7*$B$2` and `=C7*$B$3`, where cells B2 and B3 contain the mean 35000 and standard deviation 25000 of the project's total profits. Then individual 1's certainty equivalent is calculated in cell F7 by the formula `=D7-(0.5/B7)*E7^2`, where B7 contains individual 1's risk tolerance 20000. Individual 2's share is calculated by `=1-C7` in cell C8, and copying D7:F7 to D8:F8 yields individual 2's certainty equivalent for her share in cell F8. Cell F11 computes the sum of the individuals' certainty equivalents by the formula `=SUM(F7:F8)`.

Now we can use Solver in this spreadsheet to maximize the sum of the computed certainty equivalents in cell F11 by changing individual 1's percentage share of the project in cell C7. The result is that Solver returns the value 0.4 in cell C7, as shown in Figure 8.1. When individual 1 takes a 40% share, his expected monetary value is 0.40*35000 = $14,000 and his standard deviation is 0.40*25000 = $10,000, and so his certainty equivalent is

CE(1) = 14000 − (0.5/20000)*(10000^2) = 14000 − 2500 = $11,500

When individual 2 takes a 60% share, her expected monetary value is 0.60*35000 = $21,000 and her standard deviation is 0.60*25000 = $15,000, and so her certainty equivalent is

CE(2) = 21000 − (0.5/30000)*(15000^2) = 21000 − 3750 = $17,250

When they plan to share the risks in this way, their total certainty equivalent of the project is

CE(1) + CE(2) = 11500 + 17250 = $28,750

This total $28,750 is the maximal sum of certainty equivalents that the partners can achieve by sharing the profits of this project.

In this optimal sharing rule, the ratio of 2's share to 1's share is 0.6/0.4 = 1.5. Notice that the ratio of 2's risk tolerance to 1's risk tolerance is exactly the same 30000/20000 = 1.5. This result is not a coincidence, as the following general fact asserts.

Fact 1. Suppose that a group of individuals have formed a partnership to share the risky profits from some joint venture or gamble, and each individual j in this group has a constant risk tolerance that we may denote by $r_j$. Let R denote the sum of all the partners' risk tolerances ($R = \sum_j r_j$). Then these individuals can maximize the sum of their certainty equivalents by sharing the risky profits among themselves in proportion to their risk tolerances, with each individual j taking the fractional share $r_j/R$ of the risky profits.

For this example, Fact 1 yields the same optimal shares that Solver returned in Figure 8.1. The sum of the partners' risk tolerances here is

$$R = r_1 + r_2 = 20000+30000 = \$50,000.$$

So the optimal share for individual 1 is 20000/50000 = 0.4, the same share that Solver generated in cell C7.

For any such partnership, we may define the total risk tolerance of a partnership to be the sum of the risk tolerances of the individual partners. For this example, we have seen that the partnership's total risk tolerance is R = \$50,000. Now, if we considered the partnership as a corporate person with constant risk tolerance equal to this total R, then a Normal lottery with mean \$35,000 and standard deviation \$25,000 would have certainty equivalent

$$\mu - (0.5/R)*\sigma\text{^}2 = 35000 - (0.5/50000)*(25000\text{^}2) = \$28{,}750$$

for this partnership, as is calculated in cell C11 of Figure 8.1. Notice that this corporate certainty equivalent is exactly the same as maximized sum of the partners' individual certainty equivalents in cell F11 under the optimal sharing rule. The following general fact asserts that this result is also not a coincidence.

Fact 2. Consider a group of individuals who have formed a partnership to share the risky profits from some joint venture or gamble, where each individual has constant risk tolerance, as assumed in Fact 1. Let R denote the sum of all the partners' individual risk tolerances ($R = \sum_j r_j$). Then the maximal sum of the partners' certainty equivalents that can be achieved by optimal risk sharing (as described in Fact 1) is equal to the certainty equivalent of the whole gamble to an individual who has a constant risk tolerance equal to the sum of these partners' risk tolerances. Thus, to maximize the sum of their certainty equivalents, the partnership should evaluate gambles according to its total risk tolerance, whenever the partners have a choice about which gambles to undertake.

Facts 1 and 2 here do not require the gamble to be Normal. We have used the special formula for certainty equivalents of Normal gambles, but the same results can also be obtained for more general distributions. It would only be more difficult to compute the certainty equivalents....

Fact 2 can give us some sense of why businesses are typically more risk tolerant than individuals, because the risks of a business may be shared among many investors. When shares of a company are owned by 50 people whose average risk tolerance is \$20,000, then Fact 2 asserts that the company itself should evaluate risks with a risk tolerance of \$1,000,000. Fact 1 tells us that, among these 50 people, the ones with greater risk tolerance should have a greater share of the company.

The above discussion assumes that partners should want to maximize the sum of their certainty equivalents. This is a good assumption, but it needs some defense. After all, any single partner may care only about his own certainty equivalent of what he gets from the partnership. Why should anyone care about maximizing this sum of all certainty equivalents? The answer is given by the following fact.

Fact 3. Consider a risk-sharing partnership where all partners have constant risk tolerance. If the partners were planning to share risks according to a sharing rule that does not maximize the sum of the partners' certainty equivalents, then any partner j could propose another sharing rule rule that would increase j's own certainty equivalent and would not decrease the certainty equivalents of any other partners.

To understand Fact 3, notice first that adding any fixed payment from one partner to another partner would not change the sum of the partners' certainty equivalents. A net payment of x dollars from partner 2 to partner 1 (when there is no uncertainty about this amount x) would decrease 2's certainty equivalent by x and would increase 1's certainty equivalent by x, because each partner is assumed to have constant risk tolerance. Thus the net payment of x dollars would leave the sum of their certainty equivalents unchanged.

Now, suppose that the partners were originally planning to use some sharing rule does not maximize the sum of the partners' certainty equivalents. Then consider any other sharing rule that is optimal, in the sense of maximizing the sum of the partners' certainty equivalents. Changing to this "optimal" sharing rule would increase some partners' certainty equivalents, but it might also decrease other partners' certainty equivalents. But let us now modify this optimal rule by adding some net payments that will cancel out these changes for all partners except one, say partner j. Any partner whose certainty equivalent would decrease should receive an additional payment equal to the amount of his decrease, to be paid by this partner j. Any other partner whose certainty equivalent would increase should make an additional payment equal to the amount of his increase, paying it to partner j. So when these payments have been added into the optimal sharing rule, everybody other than partner j is getting exactly the same overall certainty equivalent as under the original plan. But adding these fixed payments does not change the sum of the partners' certainty equivalents. So our modified optimal plan (with the additional payments) still maximizes the sum of the partners' certainty equivalents, and so it must generate a strictly greater sum of certainty equivalents than the original plan. Thus, with everybody else's certainty equivalent unchanged, partner j must be enjoying a strictly greater certainty equivalent under this new plan. This proves Fact 3.

Fact 3 tells us that it is always optimal for partners to maximize the sum of their certainty equivalents. To apply Fact 3, consider our sharing example from the perspective of individual 1, in a situation where the option to buy and develop the land was originally his alone, and so he has the option to undertake the project without any participation from individual 2. Individual 2, of course, has the alternative of not participating in the project, in which case she would get $0. Any sharing rule that gives 2 a certainty equivalent more than $0 would be better for her than nonparticipation, and so could be accepted by her. The best possible sharing rule for individual 1 would be one that maximizes 1's certainty equivalent subject to the constraint that 2's certainty equivalent should not be less than $0. Fact 3 tells us that this can be achieved by sharing in the optimal proportions, to maximize the sum of the individuals' certainty equivalents, with an additional payment from individual 2 to individual 1 that reduces 2's certainty equivalent to $0 (or to some value slightly greater than $0). By Fact 1, the optimal share for individual 2 is 60% of this project, because 30000/(20000+30000) = 0.6, and we have seen that a 60% share with no additional payment would have certainty equivalent $17,250 to individual 2 (see cell F8 in Figure 8.1). So the best possible sharing rule for individual 1 would be to sell individual 2 a 60% share of this project for an initial payment of $17,250 (or slightly less than this), which just exhausts 2's perceived gains from participating in the partnership. After selling 60% of the investment to individual 2 for this maximal price, individual 1 would have $17,250 in cash plus a risky investment that is worth $11,500 to him (his certainty equivalent for a 40% share). Thus, selling 60% to individual 2 for $17,250 would make individual 1's overall certainty equivalent from the project 17250 + 11500 = $28,750. This is the most he could possibly hope for in any sharing rule, because it allocates to him all the maximal sum of certainty equivalents that the two partners can get from this project.

Of course, individual 2 would prefer to pay less than $17,250 for a 60% share, and she might try to negotiate for a lower price in this situation. Recall that $19,375 was 1's certainty equivalent for undertaking the project himself, and so 1 would not accept any certainty equivalent less than $19,375 when his alternative is owning 100% of the project himself. Because 1's certainty equivalent for 40% of the project is $11,500, he needs an additional payment of 19375−11500 = $7875 to raise his certainty equivalent to this level. So the best possible sharing rule for individual 2 here would be for her to buy 60% of the project (her optimal share) for just a bit above $7875, which is the lowest price that individual 1 would be willing to accept.

But regardless of who initially owns the project, the partners can can agree that they should maximize the sum of their certainty equivalents by sharing the risky returns in proportion to their risk tolerances. How this maximal value is divided among them is a bargaining problem. If one of them initially owns more than his or her optimal share of the project, there will exist a range of transfer prices at which the individuals could both gain by changing to their optimal shares. In this situation, the price that individual 2 may actually pay to buy 60% of the project must be a question of bargaining between the two individuals, and without a theory of bargaining we can only say here that it should be somewhere between $7875 and $17,250.

Facts 1, 2, and 3 here require the assumption that all partners have constant risk tolerance, but they do not require any assumption about the probability distribution from which the partnership's profits will be drawn. Normality here was only used to compute exact certainty equivalents in Figure 8.1. ...In Figure 8.1, when we asked Solver to find an optimal sharing rule, we implicitly assumed that the two partners would share the profits linearly. Here a partner's share is <u>linear</u> if each extra dollar of profit would increase the partner's income by the same amount. But this linearity assumption is not necessary. Even when we allow that a partner's income may be a nonlinear function of the total profit earned, the linear sharing rule that we described in Fact 1 is still optimal for maximizing the sum of the certainty equivalents among partners who all have constant risk tolerance.