ECONOMICS 30300a ASSIGNMENT 3b: Adverse-selection market exercise [*due May 4, 2018*]

<u>3.</u> Consider a population of risk-averse consumers who face uncertain but potentially insurable losses that will be independently drawn from normal distributions. Each consumer has constant risk tolerance, with the risk-tolerance index T=10, and each consumer's loss will be drawn from a normal distribution with standard deviation σ=20. But the expected value of this loss will depend on the consumer's risk type, which may be high or low. A consumer of the low-risk type has an expected loss of m_L=40, but a consumer of the high-risk type has an expected loss of m_H=60. In the overall population, μ(L)=80% of consumers are the low-risk type and μ(H)=20% of consumers are the high risk type. Each consumer knows his own risk type, but insurance companies have no way to directly observe anyone risk type (other than by the different choices that they might make in the market). Consider linear insurance contracts which are parameterized by the fraction x of the consumer's loss that the insurance company will cover, where 0≤x≤1. That is, an x-contract specifies that the insurance company will pay x$\tilde{S}$ to the consumer when the consumer's risky loss turns out to be $\tilde{S}$. (With normally distributed losses, there is a small probability that a consumer's "loss" $\tilde{S}$ might be negative, in which case |$\tilde{S}$| would actually denote a risky income for the consumer, of which the consumer would be contractually obligated to pay the fraction x|$\tilde{S}$| to the insurance company.)

(a) Find the coverage fraction ȳ such that a high-risk consumer would be indifferent between [1] buying full insurance (x=1) at a price such that risk-neutral insurers would expect to just break even in selling to high-risk consumers (C(1,H)) and [2] buying a ȳ-contract at a price such that risk-neutral insurers would expect to just break even in selling to low-risk consumers (C(ȳ,L)).

(b) Construct a competitive equilibrium of this insurance market, when competitive insurance companies can sell linear insurance contracts for any fraction x between 0 and 1. You should specify equilibrium prices of x-contracts, for all x in the interval [0,1], and you should identify the contracts that will be purchased by each type of consumer. Then verify that this equilibrium satisfies the equilibrium conditions that we introduced in class (the simplified version of Azevedo & Gottlieb's equilibrium concept). Also, compute the certainty-equivalent value Ū(q,t) for each type of consumer in this equilibrium.

(c) Now consider what contract the low-risk consumers would want to buy if all contracts could be priced so that risk-neutral insurers would expect to just break even in selling to the overall population (80% low-risk types and 20% high-risk types). Find the coverage fraction y* that would maximize the certainty equivalent for a low-risk consumer in such an ideal world where contracts could be priced according to their expected cost with pooling of all consumers. Show that both the low-risk types and the high-risk types would strictly prefer to buy the contract y* at this pooling price ($\sum_{t \in \{L,H\}} \mu(t)C(y^*,t)$) over the contracts that they buy with the equilibrium prices that you found in part (b).

(d) Find the coverage fraction ŷ such that a low-risk consumer would be indifferent between [1] buying a ŷ contract at the price such that risk-neutral insurers would expect to just break even in selling ŷ-contracts to low-risk consumers (C(ŷ,L)) and [2] buying the y* contract (from part (c)) at the pooling price such that risk-neutral insurers would expect to just break even in selling y*-contracts to the whole population.

(e) Now suppose that the government banned the sale of insurance contracts with any coverage fraction x such that ŷ < x < y*. Show that there is now a competitive equilibrium with pooling at y*. Verify all our conditions for an equilibrium, and show that this equilibrium makes all consumers strictly better off than the competitive equilibrium that you found in part (b).

ECONOMICS 30300a ASSIGNMENT 3a: Moral-hazard exercises [*due Friday May 4, 2018*]

1. Observable revenue **y** will be drawn from a Normal probability distribution with mean m_H if the agent chooses high effort, but with mean m_L if the agent chooses low effort. The variance will be σ^2 regardless of the agent's effort. The agent's monetary cost of effort is c_H if his effort is high, or c_L if his effort is low. Suppose $c_H > c_L$ and $m_H > m_L$. The agent has a constant risk tolerance T. Let w_0 denote the risk-free wage that the agent could get from other employers (with zero effort costs).

(a) Suppose that agent's wage must depend on observable profit **y** only by a linear formula $w(\mathbf{y}) = \alpha + \beta \mathbf{y}$. We discussed in class how to define the optimal linear wage formula to minimize a risk-neutral principal's expected wage cost subject to the constraint of giving the agent an incentive to participate with high effort. Given the same participation and moral-hazard constraints, show how the principal's optimal wage plan (maximizing the principal's expected utility or net certainty equivalent subject to the agent's participation and moral-hazard constraints) could change if the principal instead had some constant risk tolerance S. (*Hint:* You should find that the answer depend on whether S is greater or less than some critical value $\hat{S}$.)

(b) For a risk-neutral principal who wants to maximize expected net profit $E(\mathbf{y} - w(\mathbf{y}))$, write down the inequalities that these parameters must satisfy for the principal to prefer motivating high effort (with optimal wages) over motivating low effort or not hiring the agent at all (and consequently getting zero revenue).

(c) Now consider the special case of $c_H = 1$, $c_L = 0$, $m_H = 2$, $m_L = 0$, $\sigma = 2$, $w_0 = 0$, T=2. Compute the minimal expected wage cost and the expected net profit for the risk-neutral principal with the optimal linear wage for motivating high effort.

(d) For the parameters in (c), instead of using a linear formula to determine the wage as a function of the revenue **y**, suppose that the principal will use a three-wage rule, paying some (low) wage w_L if $\mathbf{y} \le -1$, some (middle) wage w_M if $-1 < \mathbf{y} \le 1$, and some (high) wage w_H if $\mathbf{y} > 1$. Find the three-wage rule (w_L, w_M, w_H) that minimizes the expected wage cost for the principal subject to the participation constraint and the incentive constraint that the agent is be willing to work here with high effort. First show a system of equations that specify both how these three wages would depend on the Lagrange multiplier of the moral-hazard incentive constraint and how this Lagrange multiplier can be computed. Then, with a computer, numerically estimate this Lagrange multiplier, compute the optimal three-wage rule, and compute the principal's expected costs under this optimal three-wage rule. [Hint: A standard normal cumulative distribution Φ has $\Phi(-1.5) = 0.0668$, $\Phi(-0.5) = 0.3085$, $\Phi(0.5) = 0.6915$]

2. Consider a production process which must be managed by an agent, who can manage it correctly or mismanage it. The production process begins with the investment of resources that are worth some amount I, and the output of the process (if any) will be realized next period.
Mismanagement here means that the agent secretly diverts 20% of the invested resources ($\gamma I = 0.2I$) to his private consumption. The principal cannot directly observe such mismanagement, but it significantly reduces the investment's probability of success.
One period after the investment of I, the process will either yield some positive output AI that is proportional to the size of the original investment. The probability of success is $p_H = 0.80$ if the agent manages the investment correctly, but the probability of success would become $p_L = 0.30$ if the agent mismanaged the investment. Here we may let the production proportionality constant be A=1.
Also, let $\underline{I} = 1$ denote the minimal level of investment (in $millions) for this production process.
Let π denote the price of output. Suppose that agents and investors are all risk neutral and discount

future period returns by the discount factor of $\delta=0.9$ per period. Agents have limited liability and so cannot every be paid less than 0.

(a) Suppose first that an agent's outside option is also 0 ($w_0=0$), and agents have no collateral (C=0). What is the lowest output price π at which investors are just willing to contract with an agent to manage such investments for one period? Under a contract that will just expect yield the required rate of return $1/\delta$ for investors, what will the agent be paid if his project succeeds? When the agent is hired, what is the agent's expected discounted value of his second-period rewards?

(b) Under the optimal contract that you found in part (a), what would be the investors' expected rate of return if the agent actually mismanaged the investment? (Hint: It would be much less that $1/\delta$.)

(c) In part (a), you found that the agent's expected payoff is strictly better than his outside option ($w_0=0$). What is the least amount of collateral C (to be forfeited if the project fails) that investors would need to demand from the agent at the minimal investment level $\underline{I}=1$) so that the agent could be given an incentive to manage the investment correctly without gaining an expected surplus over his outside option ($w_0=0$)? With the output price π that you found in part (a), what would be the investor's expected rate of return from their optimal contract with an agent who could offer such collateral for an investment of size $\underline{I}=1$?

(d) Now suppose that investors can hire an agent to manage this production process in up to two periods, with a contract that can specify some larger investment amount ($I_2 > \underline{I}$) for the agent to manage in the second period if his first-period production is a success, and some final payment w_3 in the third period if agent's production is successful in both periods. Suppose again that the agent cannot have any collateral for the investments that he will manage (C=0). What is the lowest output price π^* at which investors would be just willing to contract with an agent to manage such investments for T=2 periods? (Assume that the output from both periods will be sold at this constant price π^*.) Describe the efficient contract (specifying the I_2 and w_3 to follow $I_1=\underline{I}$) that would motivate the agent correctly and enable the investors to get expected discounted future returns which cover their investment costs.

(e) Now suppose that the market for this output has an inverse demand curve of the form $\pi = \max\{5-Q,0\}$, when total output Q is measured in aggregate units that are thousands of times larger than the output of an investment of $\underline{I}$. Suppose that, in period t=1, all productive investments will be managed by young agents who have just been hired in the first period of their two-period contracts. In period t=2, all productive investments will be managed by old agents in the second period of their contracts, but only a p_H fraction of the previously young agents will succeed and manage such investments in their second periods. Thus, the aggregate output quantities Q_2 and Q_3 that will be realized in periods 2 and 3 (from investments in periods 1 and 2 respectively) must satisfy $Q_3/Q_2 = p_H I_2/I_1$. The prices at which the outputs in these periods will be sold must satisfy $\pi_2 = \max\{5-Q_2,0\}$ and $\pi_3 = \max\{5-Q_3,0\}$. With a computer, find prices and aggregate output quantities π_2, π_3, Q_2, and Q_3 that are compatible with these conditions of aggregate demand no expected surplus returns for competitive investors with agents hired under the optimal two-period contract. Show also that, if the prices and quantities π_t and Q_t continue to cycle, with all even periods looking like period 2, all odd periods looking like period 3, and all new agents hired to start their first period of production only in odd periods (like period 1), then investors could expect only normal rates of return ($1/\delta$) from optimal contracts with new young agents in any period.

ECONOMICS 30300a ASSIGNMENT 2. [*due Friday April 20, 2018*]

<u>1.</u> For any given utility weights $\lambda_1>0$ and $\lambda_2>0$ and risk tolerances $T_1>0$ and $T_2>0$, derive the sharing rules (x_1,x_2) that maximize $\lambda_1E(u_1(x_1(\mathbf{Y})))+\lambda_2E(u_2(x_2(\mathbf{Y})))$ subject to the constraint $x_1(\mathbf{Y})+x_2(\mathbf{Y})=\mathbf{Y}$, when $u_1(x)=-EXP(-x/T_1)$ and $u_2(x)=-EXP(-x/T_2)$.
(You will need to find the formula for how $x_1(0)$ depends on λ_1, λ_2, T_1, and T_2.)

<u>2.</u> Individuals 1 and 2 are a brother and sister who have inherited equal shares of their mother's scattered real estate properties. They have constant risk tolerances $T_1=\$20,000$ and $T_2=\$45,000$ respectively. Appraisals of the property by independent real-estate experts suggest that the total returns from selling the properties individually over a period of months may be viewed as a draw from a Normal distribution with mean $200,000 and standard deviation $75,000.
But one real estate agent just offered to buy all the properties right now for $150,000, provided that they accept the offer within 24 hours.
(a) Show that, if they were constrained to share equally all income from the sale of the properties, then 1 and 2 would disagree about whether the Normal($\mu=\$200,000$, $\sigma=\$75,000$) gamble was preferable to the certain $150,000.
(b) Suppose that, because of their disagreement, the real-estate agent's 24-hour offer has been lost, and so their prospective returns now are defined by the Normal($\mu=\$200,000,\sigma=\$75,000$) distribution. Suppose 2 can make an offer acquire any part of her brother's 50% share, for any price now that would be acceptable to her brother. Find the best offer for 2, to maximize her expected utility, subject to the constraint that 1 should be willing to accept the offer. That is, what share should 2 try to acquire, and what is the smallest payment that 1 would accept now in exchange for this reduction of his share of the risks? What is the maximum amount that 2 would be willing to pay to acquire this increased share (above the 50% that she already owns)?
(c) Show that, after their shares have been adjusted as in part (b), if the real-estate agent extended the $150,000 offer for another 24 hours, both 1 and 2 would want to turn it down.
What is the smallest price for the entire estate that they would be willing to accept?

<u>3.</u> Consider again the sender-receiver example from the notes:

	$c_2=x$	$c_2=y$	$c_2=z$	
$t_1=1a$	2, 3	0, 2	−1, 0	($p(1a)=0.5$)
$t_1=1b$	1, 0	2, 2	0, 3	($p(1b)=0.5$)

It has been asserted (in section 6.7 of Myerson Game Theory) that the following μ maximizes $U_1(\mu|1a)$ among all incentive-compatible mediation plans for this example:

	$c_2=x$	$c_2=y$	$c_2=z$
$t_1=1a$	0.8	0.2	0
$t_1=1b$	0.4	0.4	0.2

(a) Which incentive constraints are binding in this mediation plan μ?
(b) Verify that this is the optimal incentive-compatible mediation plan for maximizing $U_1(\mu|1a)$ by Lagrangean analysis. It suffices to show that, with appropriate Lagrange multipliers for the binding incentive constraints, this mediation plan would maximize the sum of virtual utilities (without regard to the incentive constraints) for each reported type.

<u>4.</u> Consider a sender-receiver game with $T_1=\{a,b\}$, $C_2=\{x,y,z\}$, $p(a)=p(b)=0.5$ and (u_1,u_2) as follows:

	$c_2=x$	$c_2=y$	$c_2=z$	
$t_1=a$	3, 5	4, 4	0, 0	($p(1a)=0.5$)
$t_1=b$	2, 0	4, 4	1, 5	($p(1b)=0.5$)

Find the incentive-compatible mediation plan that maximizes the expected payoff for player 2 (the receiver).
Hints: In this mediation plan μ, the positive conditional probabilities are $\mu(x|a)>0$, $\mu(y|a)>0$, $\mu(y|b)>0$, and $\mu(z|b)>0$; and the binding incentive constraints are that 1 should not want to report "a" when $t_1=b$ and that player 2 should not want to choose z when y is recommended. (That is, $\alpha_1(a|b)>0$ and $\alpha_2(z|y)>0$.) Find the Lagrange multipliers for the binding incentive constraints that verify the optimality of this plan.

ECONOMICS 30300a ASSIGNMENT 1. [*due Friday April 6, 2018*]

<u>1.</u> Players 1 and 2 each must decide whether to fight for a valuable prize. If both players decide to fight, then they both lose $1, and nobody gets the prize (it is destroyed). If one player decides to fight but the other does not, then the player who is willing to fight gets the prize. A player who does not fight is guaranteed a payoff of 0.

Everybody knows that the prize is worth $V_2 = \$2$ to player 2. But the prize may be worth more to player 1.

Let V_1 denote the value of the prize to player 1.

In terms of V_1, the players' payoffs (u_1, u_2) will depend on their actions as follows:

Player 1: \ Player 2:	NotFight	Fight
NotFight	0, 0	0, 2
Fight	V_1, 0	−1, −1

Let us explore some different assumptions about this value V_1.

(a) Suppose first that the value of the prize to player 1 is $V_1 = \$3$, and everybody knows this. Find all equilibria of this game, including a mixed-strategy equilibrium in which both players have a positive probability of fighting.

(b) Suppose next that the value of the prize to player 1 is either $V_1 = \$2$ or $V_1 = \$3$.
Player 1 knows his actual value, but player 2 thinks each of these possibilities has probability 1/2.
Find a Bayesian equilibrium where player 2 randomizes between fighting and not fighting.

(c) Finally, suppose that the value of the prize to player 1 is $V_1 = \$2 + \tilde{t}_1$ where $\tilde{t}_1$ can be any number between 0 and 1.
Player 1 knows its actual value, but player 2 thinks of $\tilde{t}_1$ as a uniform random variable on the interval from 0 to 1. Find a Bayesian equilibrium where player 2 randomizes between fighting and not fighting.

<u>2.</u> Consider a two-person game where player 1 chooses T or B, and player 2 chooses L or R.
When they play this game, player 2 also knows whether her type is A or B.
The players' utility payoffs (u_1, u_2) depend on their actions and on player 2's type as follows:

2'sType = A	L	R		2'sType = B	L	R
T	4,0	0,2		T	4,0	0,4
B	0,4	2,0		B	0,2	2,0

(a) Suppose first that player 1 thinks that 2's type is equally likely to be A or B. Find a Bayesian equilibrium.
(b) How would the Bayesian equilibrium change in a game where player 1 thinks that player 2 has probability 1/6 of being type A, and has probability 5/6 of being type B?

<u>3.</u> Consider a game with n players. A public good that is worth V to each player will be supplied if at least one player volunteers to provide it. (Think of the public good as notifying the fire department about a fire.)
Each player i has a privately known cost of volunteering t_i that is an independent random variable drawn from a uniform distribution on the interval from 0 to c. So with type t_i, player i's utility payoff is $V−t_i$ if player i volunteers, is V if player i does not volunteer but at least one other player volunteers, and is 0 if nobody volunteers.
When they decide whether to volunteer, each player i knows only his own cost-type t_i.
Given the parameters (n,V,c) satisfying n≥2 and V>c>0, characterize a symmetric Bayesian equilibrium of this game. You may characterize the equilibrium in terms of a quantity that satisfies some equation, which you should write as simply as you can, and then solve this equation numerically for the case of n=4, V=100, and c=1.

<u>4.</u> Consider the following Bayesian game, where player 2's type is her vulnerability at (T,L).
Player 2 knows the amount t_2 that she would have to pay player 1 if they play (T,L), but player 1 only knows that t_2 was drawn from a Uniform distribution on the interval from 0 to 1.

Player 1: \ Player 2:	L	R
T	t_2, −t_2	0, 0
B	0, 0	0.5, −0.5

(a) A student said, "player 2 should choose R if $t_2 > 0.5$, but 2 should choose L if $t_2 < 0.5$". Show that this student's analysis is not compatible with any equilibrium, by first computing player 1's best response to the strategy that the student has recommended for 2, and then computing player 2's best response to this best-response of player 1.
(b) Find a Bayesian equilibrium of this game. Be sure to fully specify the strategies for both players.