Summary Overview of Topics in Econ 30200b:
Decision theory: strong and weak domination by randomized strategies, domination theorem, expected utility theorem (consistent decisions under uncertainty should maximize the expected value of some vonNeuman-Morgenstern utility).
Finite strategic-form games: rationalizable strategies after iterative elimination of strongly dominated strategies, best-response functions, finding an equilibrium or all equilibria of a small (2×2 or 2×3) game (finding the support of an equilibrium, checking the complementary slackness conditions), finding equilibria with particular kinds of supports, finding symmetric equilibria of symmetric games which may be larger ($2\times2\times...\times2$, 3×3).
Games where players choose numbers subject to bounds: find best-response functions and (pure-strategy) Nash equilibria using first-order and boundary conditions on derivatives $\partial u_i/a_i$.
Extensive-form games (first with perfect information, then general information sets): strategies, mixed strategies and behavioral strategies, the normal representation in strategic form and its Nash equilibria, subgame-perfect equilibria, sequential equilibria (move probabilities, prior probabilities of nodes, belief probabilities, consistency of beliefs, sequential rationality of strategies, identifying beliefs at zero-probability information sets), finding a sequential equilibrium with a particular kind of support.
Handling discontinuous strategies in subgame-perfect equilibria of games with perfect information where players choose numbers (infinitely many possible moves).
Repeated games: maximizing δ-discounted sum of payoffs; characterizing behavioral strategy profiles (scenarios) with a set of social states, state-dependent strategies, and a state-transition rule; recursive formula for computing state-dependent δ-discounted values; one-deviation conditions for a subgame-perfect equilibrium.

ECONOMICS 30200b ASSIGNMENT 5 *[not to be handed in, answers will be posted]*

1. Consider a repeated game where 1 and 2 repeatedly play the game below infinitely often.

	a_2	b_2
a_1	8, 8	1, 2
b_1	2, 1	0, 0

The players want to maximize their δ-discounted sum of payoffs, for some $0 \leq \delta < 1$.
Consider the following state-dependent strategies: The possible states are state 1 and state 2.
In state 1, we anticipate that player 1 will play b_1 and player 2 will play a_2.
In state 2, we anticipate that player 1 will play a_1 and player 2 will play b_2.
The game begins at period 1 in state 1. The state of the game would change after any period where the outcome of play was (a_1,a_2), but otherwise the state always stays the same.
What is the lowest value of δ such that these strategies form a subgame-perfect equilibrium?

2. Consider a repeated game where 1 and 2 repeatedly play the game below infinitely often.

	a_2	b_2
a_1	3, 3	0, 5
b_1	5, 0	−4, −4

The players want to maximize their δ-discounted sum of payoffs, for some $0 \leq \delta < 1$.
(a) Find the lowest value of δ such that you can construct an equilibrium in which the players will actually choose (a_1,a_2) forever, but if any player i ever chose b_i at any period then they would play the symmetric randomized equilibrium of the one-stage game forever afterwards.
(b) What is the lowest value of δ such that you can construct a subgame-perfect equilibrium in which the players will actually choose (a_1,a_2) forever, but if some player i unilaterally deviated to b_i at any period then that player i would get payoff 0 at every round thereafter? Be sure to precisely describe state-dependent strategies that form this equilibrium.

3. Consider a repeated game where 1 and 2 repeatedly play the game below infinitely often.

	a_2	b_2
a_1	0, 8	2, 0
b_1	8, 0	0, 2

Each player i wants to maximize his or her δ_i-discounted sum of payoffs, for some δ_1 and δ_2, where each $0 \leq \delta_i < 1$.
Find the lowest values of δ_1 and δ_2 such that you can construct an equilibrium in which the players will actually alternate between (a_1,a_2) and (b_1,a_2) forever, but if any player ever deviated then they would play the randomized equilibrium of the one-stage game forever afterwards.

ECONOMICS 30200b ASSIGNMENT 4 *[Due Mar 6, 2018]*

1. The new widget production process that firm 1 is developing is equally likely to have high cost or low cost. Firm 1 will learn whether its production cost is high or low at the beginning of next year. Then firm 1 can choose whether to build a new factory or not. Firm 2 will not be able to observe firm 1's production cost, but firm 2 will be able to observe whether firm 1 builds a new factory or not. Firm 2 will subsequently decide whether to enter the widget market or not. Firm 2 will earn \$2 million (in present discounted value of long-run profits) from entering the widget market if firm 1's production cost is high, but firm 2 will lose \$4 million from entering if firm 1's production cost is low. (These payoffs are relative to a payoff of \$0 to firm 2 if it does not enter.) Let a payoff of 0 to firm 1 denote its profit if new cost is high, firm 1 does not build, and firm 2 does not enter. Lower costs in the new process will increase firm 1's profit by \$4 million (ceteris paribus). Building a new factory would add \$2 million more to firm 1's profit if the new process has low cost (because conversion to the new process would be easier in a new factory), but building a new factory would subtract \$4 million from firm 1's profit if the new process has high cost. In any event, firm 2's entry into the widget market would reduce firm 1's profit by \$6 million. Both firms are risk neutral.
(a) Describe this game in extensive form.
(b) Construct the normal representation of this game in strategic form (the normal form).
(c) Analyze this strategic-form game by iterative elimination of weakly dominated strategies.
(d) Find two different pure-strategy equilibria of this strategic-form game. For each, show the beliefs (if any) that would make it a sequential equilibrium of the extensive-form game.

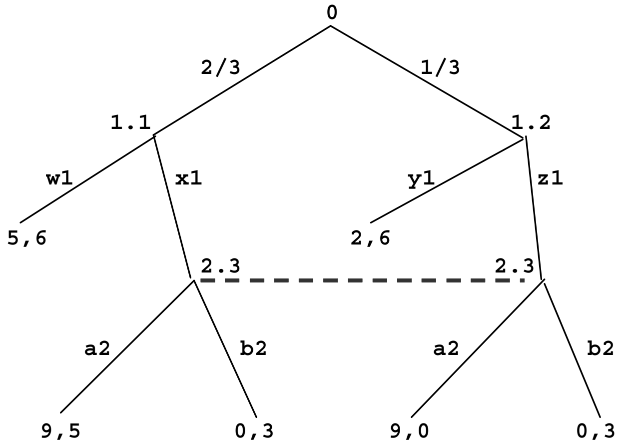
2. Consider a game where player 1 must choose T or B, player 2 must choose L or R, and their payoffs depend on their choices as follows.

Player 1 \ Player 2:	L	R
T	3, 2	1, 1
B	4, 3	2, 4

Suppose that player 1 moves first, and then player 2 makers her choice after observing 1's move.
(a) Show the extensive-form game with perfect information that describes this situation.
(b) Show the normal representation in strategic form for the extensive-form game in part (a).
(c) Find the unique sequential (subgame-perfect) equilibrium of this game.
(d) Find a Nash equilibrium of this game that is **not** sequentially rational (or subgame-perfect).

3. Consider again the game in the previous exercise 2 where player 1 moves first. But now suppose that, whatever 1 chooses, the probability that player 2 will correctly observe 1's action is 0.9, and there is probability 0.1 that player 2 will mistakenly observe the other action (which 1 did not choose). The payoffs depend on the players' **actual** choices according to the previous table (so, for example, if 1 chose T but 2 mistakenly observed B and chose R then 2's payoff would be 1).
(a) Show the extensive-form game that describes this situation.
(b) Show the normal representation in strategic form for the extensive-form game in part (a).
(c) Find a sequential equilibrium in which player 2 would choose [L] for sure if she observed T.
*(d) Characterize the other sequential equilibria of this game.

4. Consider the following extensive-form game, where player 1 observes the chance move, but player 2 does not observe it. If 2 gets to move, she knows only that 1 chose either x1 or z1.



(a) Find a sequential equilibrium in which the (prior) probability of player 2 getting to move is 1.

(b) Find a sequential equilibrium in which the probability of player 2 getting to move is 0.

(c) Find a sequential equilibrium in which the probability of player 2 getting to move is strictly between 0 and 1.

5. Players 1 and 2 are in a sequential all-pay-own-bid auction for a prize worth \$3. First, player 1 must pay \$1 or pass. When anyone passes, the other player gets the \$3 prize (and game ends). Otherwise, the other player can bid next, and must either pay \$2 (if he has it) or pass.
A player cannot pay more than his given available funds. This game has perfect information.

(a) Find a subgame-perfect equilibrium if each player has \$4 available to spend.

(b) Find a subgame-perfect equilibrium if each player has \$5 available to spend.

6. Player 1 chooses a_1 between 0 and 1 ($0 \leq a_1 \leq 1$), and player 2 also chooses a_2 between 0 and 1 ($0 \leq a_2 \leq 1$). Their payoffs (u_1,u_2) depend on the chosen numbers (a_1,a_2) and a known parameter γ as follows:

$$u_1(a_1,a_2) = \gamma\, a_1 a_2 - (a_1)^2\,,$$
$$u_2(a_1,a_2) = 2a_1a_2 - a_2.$$

(a) Given γ=1.5, find all (pure) Nash equilibria of this game if players choose their a_i independently.

(b) Given γ=0.8, find all (pure) Nash equilibria of this game if players choose their a_i independently.

(c) Given γ=1.5, find a subgame-perfect equilibrium of this game if player 1 chooses a_1 first, and then player 2 chooses a_2 after observing a_1.

(d) Given γ=0.8, find a subgame-perfect equilibrium of this game if player 1 chooses a_1 first, and then player 2 chooses a_2 after observing a_1.

ECONOMICS 30200b ASSIGNMENT 3 *[Due Feb 23, 2018]*

1. Find all Nash equilibria of the following 2×3 game:

Player 1: \ Player 2:	L	M	R
T	0, 4	5, 6	8, 7
B	2, 9	6, 5	5, 1

2. Consider the following 3×3 games that depend on a parameter α:

Player 1: \ Player 2:	L	M	R
T	α, α	$-1, 1$	$1,-1$
C	$1,-1$	α, α	$-1, 1$
B	$-1, 1$	$1,-1$	α, α

(a) Suppose we are given $\alpha>1$. Show that there are equilibria where the support includes two pure strategies for each player. Show also that there are pure-strategy equilibria, and show that there is an equilibrium where the support includes all three pure strategies for both player.
(b) For the support sets that you found in part (a), which of them also can be the support of an equilibrium when $\alpha<1$?
(c) Suppose $\alpha=0$, but now change the game by eliminating player 1's option to choose B.
Find all equilibria of this 2×3 game.

3. Consider an all-pay-own-bid auction among n bidders. Each bidder i independently chooses a nonnegative bid c_i, which he will pay in the auction regardless of whether he wins or not, but if he is the high bidder (with $c_i > c_j\ \forall j\neq i$) then he will win a prize worth $V>0$. (So $u_i = V-c_i$ if i wins, else $u_i = -c_i$.)
Find a symmetric equilibrium in which each bidder randomizes over the interval from 0 to V.
In this symmetric randomized equilibrium, what is the expected value of each bidder's bid c_i?

4. Firms 1 and 2 are competing in the same market. Each firm i must choose a quantity q_i to supply, and the market price p will depend on their choices according to the inverse demand formula $p(q_1,q_2) = \max\{A-(q_1+q_2), 0\}$. The total cost of production for each firm i is $(q_i)^2$, and so the total profit for firm i will be $u_i(q_1,q_2) = p(q_1,q_2)\, q_i - (q_i)^2$.
(a) For any given q_2, what would be firm 1's best response q_1 to maximize u_1?
(b) Find the Nash equilibrium of this game when the two firms choose their supply quantities simultaneously and independently. Compute each firm's expected profit in this equilibrium.
*(c) Now suppose that firm 1 chooses q_1 first, and then firm 2 chooses its q_2 after observing q_1.
Find the subgame-perfect equilibrium of this game with perfect information, and compute each firm's expected profit in this equilibrium.

ECONOMICS 30200b ASSIGNMENT 2 *[Due Feb 20, 2018]*

1. Consider a game where players 1 chooses an action in {T,B}, player 2 simultaneously chooses an action in {L,R}, and their payoffs (u_1,u_2) depend on their actions as follows:

Player 1 \ Player 2:	L	R
T	1, 9	8, 3
B	7, 2	4, 5

Find all Nash equilibria of this game (including equilibria with randomized strategies), and compute the players' expected payoffs in each equilibrium.

2. Consider a game where players 1 chooses an action in {T,B}, player 2 simultaneously chooses an action in {L,R}, and their payoffs (u_1,u_2) depend on their actions as follows:

Player 1 \ Player 2:	L	R
T	0, 3	8, 5
B	4, 6	7, 2

(a) Find all Nash equilibria of this game (including equilibria with randomized strategies), and compute the players' expected payoffs in each equilibrium.
(b) How would your answer change if player 1's payoff from (B,R) were increased from 7 to 9?

3. Consider a game where player 1 must choose T or M or B, player 2 must choose L or R, and their utility payoffs (u_1,u_2) depend on their choices as follows:

Player 1 \ Player 2:	L	R
T	6, 1	4, 9
M	5, 7	6, 0
B	9, 7	1, 8

(a) Show a randomized strategy that strongly dominates T for player 1.
(b) Find an equilibrium in randomized strategies for this game, and compute the expected payoff for each player in this equilibrium.
(c) Assuming that player 2 will act according to her equilibrium strategy that you found in part b, what would player 1's expected payoff be if he chose the action T?

4. Players 1 and 2 are involved in a joint project, and each must decide whether to work or shirk. If both work then each gets a benefit worth 1, but each also has a private effort cost e of working. So their payoffs depend on their payoffs (u_1,u_2) depend on their actions as follows:

Player 1 \ Player 2:	2 works	2 shirks
1 works	$1-e$, $1-e$	$-e$, 0
1 shirks	0, $-e$	0, 0

Suppose that e is a known parameter between 0 and 1. Find all Nash equilibria of this game.

5. Consider the penalty kick in soccer. Player 1 is the kicker, and player 2 is the goalie.
Player 1 can kick to left or right. Player 2 must simultaneously decide to jump left or right.
The probability that of the kick being blocked is λ if they both go left, but is ρ if they both go right.
If they choose different directions then the probability of the kick being blocked is 0.
So the players' payoffs (u_1,u_2) depend on their choices as follows:

Player 1 \ Player 2:	L	R
L	$1-\lambda$, λ	1, 0
R	1, 0	$1-\rho$, ρ

(a) Find a Nash equilibrium, and compute the expected payoffs to each player.
(b) If player 2 becomes more skilled at defending left then λ would increase in this game.
How would this parametric change affect 2's probability of choosing left in equilibrium?

6. Find the nonrandomized Nash equilibria of the two-player strategic game in which each player's set of actions is the nonnegative real numbers and the players' payoff functions are
$u_1(c_1,c_2) = c_1(c_2-c_1)$, $u_2(c_1,c_2) = c_2(1-c_1-c_2)$.

7. Players 1 and 2 are involved in a joint project. Each player i independently chooses an effort c_i that can be any number in the interval from 0 to 1; that is, $0 \leq c_1 \leq 1$ and $0 \leq c_2 \leq 1$.
(a) Suppose that their output will depend on their efforts by the formula $y(c_1,c_2) = 3c_1c_2$, and each player will get half the output, but each player i must also pay an effort cost equal to c_i^2.
So $u_1(c_1,c_2) = 1.5c_1c_2 - c_1^2$ and $u_2(c_1,c_2) = 1.5c_1c_2 - c_2^2$.
Find all Nash equilibria without randomization.
(b) Now suppose that their output is worth $y(c_1,c_2) = 4c_1c_2$, of which each player gets half, but each player i must also pay an effort cost equal to c_i.
So $u_1(c_1,c_2) = 2c_1c_2 - c_1$ and $u_2(c_1,c_2) = 2c_1c_2 - c_2$.
Find all Nash equilibria without randomization.

8. There are two players numbered 1 and 2. Each player i must choose a number c_i in the set {0,1,2}, which represents the number of days that player i is prepared to fight for a prize that has value V=$9. A player wins the prize only if he is prepared to fight strictly longer than the other player. They will fight for as many days as both are prepared to fight, and each day of fighting costs each player $1. Thus, the payoffs for players 1 and 2 are as follows:
Player 1's payoff is $u_1(c_1,c_2) = 9-c_2$ if $c_1 > c_2$, but $u_1(c_1,c_2) = -c_1$ if $c_1 \leq c_2$.
Player 2's payoff is $u_2(c_1,c_2) = 9-c_1$ if $c_2 > c_1$, but $u_2(c_1,c_2) = -c_2$ if $c_2 \leq c_1$.
(a) Show a 3×3 matrix that represents this game.
(b) What dominated strategies can you find for each player in this game?
(c) What pure-strategy (nonrandomized) equilibria can you find for this game?
(d) Find a symmetric equilibrium in randomized strategies.

9. Consider a symmetric three-player game where each player must choose L or R.
If all three players choose L, then each of them gets payoff 1.
If all three players choose R, then each of them gets payoff 4.
Otherwise, if the players do not all choose the same action, then they all get payoff 0.
Find a symmetric randomized equilibrium in which both actions get positive probability.

10. Players 1 and 2 are bidding to buy an object in a sealed-bid auction. The object would be worth $V_1 = 53.40$ to player 1 if he could get it, but it would be worth $V_2 = 67.90$ to player 2 if she could get it. These values are commonly known by both players. Each player i chooses a bid c_i that must be a nonnegative multiple of ε, the smallest monetary unit. ($\varepsilon>0$ is given.)
The high bidder wins the object, paying the price that he or she bid, and the loser pays nothing. If their bids are equal, then they each have probability 1/2 of buying the object for the bid price.
So $u_i(c_1,c_2) = V_i-c_i$ if $c_i > c_{-i}$, but $u_i(c_1,c_2) = 0$ if $c_i < c_{-i}$, and $u_i(c_1,c_2) =0.5(V_i-c_i)$ if $c_1=c_2$.
(a) Show that, for each player i, bidding more than V_i is a weakly dominated action.
(b) Suppose that $\varepsilon=1$. Show that there is a unique nonrandomized equilibrium of this game after weakly dominated actions are eliminated, and compute the players' payoffs in this equilibrium.
(c) If we considered a sequence of games as $\varepsilon\to0$, what would be a limit of undominated equilibrium strategies and payoffs in this game? Characterize the limit of each player's bid and the limit of each player's probability of winning the object in this auction.

ECONOMICS 30200b, ASSIGNMENT 1

[These problems are not to be handed in. You should know how to solve problem 1, but problems 2 and 3 are for discussion only.]

1. A decision-maker must choose between three alternative decisions {d1,d2,d3}. Her utility payoff will depend as follows on her decision and on an uncertain state of the world in {s1,s2}:

	State s1	State s2
Decision d1	15	90
Decision d2	B	75
Decision d3	55	40

Let p denote the decision-maker's subjective probability of state s2.

(a) Suppose first that B=35. For what range of values of p is decision d1 optimal? For what range is decision d2 optimal? For what range is decision d3 optimal? Is any decision strongly dominated? If so, by what randomized strategies?
(b) Suppose now the B=20. For what range of values of p is decision d1 optimal? For what range is decision d2 optimal? For what range is decision d3 optimal? Is any decision strongly dominated? If so, by what randomized strategies?
(c) For what range of values for the parameter B is decision d2 strongly dominated?

2. A decision-maker has expressed the following preferences:
Getting $1000 for sure is as good as a lottery offering 0.27 probability of $5000 or else $0.
Getting $2000 for sure is as good as a lottery offering 0.50 probability of $5000 or else $0.
That is: [$1000] ~ 0.27[$5000]+0.73[$0], [$2000] ~ 0.50[$5000]+0.50[$0].
If this person is logically consistent, which should he prefer among the following:
a lottery offering a 0.5 probability of $2000 or else $1000 (0.5[$2000]+0.5[$1000]),
a lottery offering a 0.4 probability of $5000 or else $0 (0.4[$5000]+0.6[$0]).
Justify your answer as fundamentally as you can.

3. Members of a primitive tribe may own bundles of various goods, which anthropologists have numbered {1,..,m}. The tribe has various ritual exchange activities, numbered {1,...,n}.
In each activity j, there is a "host" and a "guest", and the host gives the guest some net quantity θ_{ij} of each good i (where a negative θ_{ij} denotes the guest giving $-\theta_{ij}$ units of i to the host).
Any tribesman may do each activity any number of times, as guest or host.
Prove a theorem of the following form: "Given any such matrix of parameters θ_{ij}, exactly one of the following two conditions is true: (1) There is a way to use some combination of these exchange activities to increase one's holdings of every good by at least one unit. (2) ...."
[You may assume that people can also do any activity j at a fractional level x_j, which would then yield a net transfer $\theta_{ij}x_j$ of each good i. But this assumption is not actually necessary.]
(*If you cannot do the proof here, at least try to formulate a conjecture as to what condition (2) might be.*)

Assignment 1 answers:

<u>1</u>(a) With B=35, d1 is optimal for $p \geq 4/7$, d2 is optimal for $4/11 \leq p \leq 4/7$, d3 is optimal for $p \leq 4/11$.
(b) With B=20, d1 is optimal for $p \geq 4/9$, d3 is optimal for $p \leq 4/9$, d2 is never optimal and is strongly dominated by q[d1]+(1−q)[d3] for $7/10 = (75-40)/(90-40) < q < (55-B)/(55-15) = 7/8$.
(c) d2 is strongly dominated when $(75-40)/(90-40) < (55-B)/(55-15)$, that is, $B < 27$.

<u>2</u> The decision-maker should prefer 0.4[$5000]+0.6[$0] over 0.5[$2000]+0.5[$1000] because by substitution and reduction: 0.5[$2000]+0.5[$1000] ~
0.5(0.27[$5000]+0.73[$0])+0.5(0.50[$5000]+0.50[$0])
~ (0.5×0.27+0.5×0.50)[$5000]+(0.5×0.73+0.5×0.50)[$0] ~ 0.385[$5000]+0.615[$0].
So the decision-maker should prefer q[$5000]+(1−q)[$0] over 0.5[$2000]+0.5[$1000]
for any $q > 0.385$, and $0.4 > 0.385$.

<u>3</u> Given any such matrix of parameters θ_{ij}, exactly one of the following two conditions is true:
(1) $\exists x \in \mathbb{R}^n$ such that $\sum_{j\in\{1,...,n\}} \theta_{ij} x_j \geq 1 \ \forall i \in \{1,...,m\}$.
(2) $\exists p \in \mathbb{R}^m$ such that $p_i \geq 0 \ \forall i \in \{1,...,m\}$, $\sum_{i\in\{1,...,m\}} p_i > 0$, $\sum_{i\in\{1,...,m\}} p_i \theta_{ij} = 0 \ \ \forall j \in \{1,...,n\}$.
For the proof, consider the closed convex set
$B = \{b \in \mathbb{R}^m \mid \exists x \in \mathbb{R}^n \text{ such that } b_i \leq \sum_{j\in\{1,...,n\}} \theta_{ij} x_j \ \forall i \in \{1,...,m\}\}$.
Statement (1) is equivalent to saying that the vector (1,...,1) is in the set B.
By the Separating Hyperplane Theorem, (1,...,1) is not in B if and only
there exists some p in $\mathbb{R}^m$ such that $\max_{b\in B} \sum_{i\in\{1,...,m\}} p_i b_i < \sum_{i\in\{1,...,m\}} p_i$.
But $\max_{b\in B} \sum_{i\in\{1,...,m\}} p_i b_i$ would be $+\infty$ if $\sum_{i\in\{1,...,m\}} p_i \theta_{ij}$ were not 0 for any j in {1,...,n}
(take x_j to $+\infty$ or to $-\infty$) or if we had any $p_i<0$ (take b_i to $-\infty$).
So the latter statement is equivalent to (2) above.

Here (2) says that the goods can be assigned prices, which are nonnegative and not all 0, such that the net exchanged value is 0 for each participant in each ritual exchange activity.

Assignment 5 answers:

1. In state 1, as long as nobody deviates, they get $U_1(b_1,a_2) = 2$, $U_2(b_1,a_2) = 1$ every period, and so their expected discounted sum of payoffs in state 1 are $V_1(1) = 2/(1-\delta)$, $V_2(1) = 1/(1-\delta)$.
In state 2, as long as nobody deviates, they get $U_1(a_1,b_2) = 1$, $U_2(a_1,b_2) = 2$ every period, and so their expected discounted values in state 2 are $V_1(2) = 1/(1-\delta)$, $V_2(2) = 2/(1-\delta)$.
In state 1, if player 1 deviates to a_1 then he expects $8+\delta V_1(2) = 8+\delta 1/(1-\delta)$,
but if player 2 deviates to b_2 then she expects $0+\delta V_2(1) = 0+\delta 1/(1-\delta)$.
So to deter deviations in state 1, we need $2/(1-\delta) \geq 8+\delta 1/(1-\delta)$ and $1/(1-\delta) \geq 0+\delta 1/(1-\delta)$
With $\delta<1$, these are equivalent to $2 \geq 8(1-\delta)+\delta 1$ and $1\geq 0(1-\delta)+\delta 1$.
The first inequality is satisfied when $\delta \geq 6/7$ and the second is satisfied for all δ between 0 and 1.
So nobody wants to deviate in state 1 if $\delta\geq 6/7$.
Similarly, nobody want to deviate in state 2 when $\delta\geq 6/7$ (but now it is player 2 who has to be deterred from deviating to a_2 which would increase her payoff to 8 now but would cause a switch back to state 1, which is worse for her). So we have a subgame-perfect equilibrium when $\delta \geq 6/7$.

In the one-stage game b_1 and b_2 are strongly dominated strategies, so the unique equilibrium is (a_1,a_2), yielding payoffs (8,8).

2. (a) In the one-period randomized equilibrium, each player i uses the randomized strategy $(2/3)[a_i]+(1/3)[b_i]$, and each player gets expected payoff equal to 2, because
$2 = (2/3)3 + (1/3)0 = (2/3)5 + (1/3)(-4)$.
Let state 0 be "cooperating", and let state 1 be "randomizing". They start in state 0.
In state 0, each player i should do a_i. They continue in state 0 as long as both do a_i,
but they change next to state 1 if anyone does b_i.
In state 1, each player i randomizes according to $(2/3)[a_i]+(1/3)[b_i]$ each period.
Once in state 1, they always continue in state 1.
The values $V_i(\theta)$ for player i of being in state θ satisfy:
$V_i(0) = 3 + \delta V_i(0)$, so $V_i(0) = 3/(1-\delta)$ for i=1,2.
$V_i(1) = 2 + \delta V_i(1)$, so $V_i(1) = 2/(1-\delta)$, for i=1,2.
Player i's discounted value of deviating to b_i in state 0 is $5+\delta 2/(1-\delta)$.
So for equilibrium, we need $3/(1-\delta) \geq 5+\delta 2/(1-\delta)$, so $3\geq 5(1-\delta)+\delta 2$, and so $\delta \geq 2/3$.

(b) We consider an equilibrium with 3 states: state 0 = "cooperate",
state 1 = "player 1 acts superior" and state 2="player 2 acts superior".
In state 0 they should play (a_1,a_2). In state 1 they play (b_1,a_2). In state 2 they play (a_1,b_2).
They start in state 0. In state 0, if they do (b_1,a_2) then they switch to state 2, but if they do (a_1,b_2) then they switch to state 1. Once in state 1 or state 2, they remain in the same state forever.

As long as equilibrium predictions are fulfilled, the state is expected to always stay the same.
So the discounted values $V_i(\theta)$ for each player i in each state θ are
$V_1(0) = 3/(1-\delta)$, $V_2(0) = 3/(1-\delta)$, $V_1(1) = 5/(1-\delta)$, $V_2(1) = 0/(1-\delta)$,
$V_1(2) = 0/(1-\delta)$, $V_2(2) = 5/(1-\delta)$.
In state 0, for each player i, the discounted value of deviating is $5 + \delta 0/(1-\delta)$.
So for equilibrium, we need $3/(1-\delta) \geq 5+\delta 0/(1-\delta)$, so $3\geq 5(1-\delta)+\delta 0$, and so $\delta \geq 2/5$.
States 1 and 2 each involve repeating forever a one-period equilibrium ((b_1,a_2) or (a_1,b_2)),
and so a player can never gain by unilaterally deviating from the equilibrium in state 1 or state 2.

3. First we must analyze the randomized equilibrium of the one-stage game.
The randomized equilibrium of the one-stage game is $(p[a_1]+(1-p)[b_1], q[a_2]+(1-q)[b_2])$ where
$Eu_1 = q(0)+(1-q)(2) = q(8)+(1-q)(0)$ and $Eu_2 = p(8)+(1-p)(0) = p(0)+(1-p)(2)$,
and so $q=0.2$, $p = 0.2$, and $Eu_1 = 1.6$, $Eu_2 = 1.6$.

The equilibrium of the repeated game has three social states:
In state 1 they play (a_1,a_2). In state 2 they play (b_1,a_2). In state 3 they play the randomized equilibrium of the one-stage game: $(0.2[a_1]+0.8[b_1], 0.2[a_2]+0.8[b_2])$.
From state 1, if (a_1,a_2) is played then next period they go to state 2,
but otherwise they go to state 3.
From state 2, if (b_1,a_2) is played then next period they go to state 1,
but otherwise they go to state 3.
From state 3, they stay in state 3 forever regardless of what anyone does.

So the discounted values $V_i(\theta)$, for each player i in each state θ, must satisfy:
$V_1(1) = 0 + \delta_1 V_1(2)$, and $V_1(2) = 8 + \delta_1 V_1(1)$.
So $(1-\delta_1^2)V_1(2) = 8$, and so we get $V_1(2) = 8/(1-\delta_1^2)$ and $V_1(1) = \delta_1 8/(1-\delta_1^2)$.
Similarly, $V_2(1) = 8 + \delta_2 V_2(2)$, and $V_1(2) = 0 + \delta_2 V_2(1)$.
and so we get $V_2(1) = 8/(1-\delta_2^2)$ and $V_2(2) = \delta_2 8/(1-\delta_2^2)$.
$V_1(3) = 1.6+\delta_1 V_1(3)$, and so $V_1(3) = 1.6/(1-\delta_1)$. Similarly, $V_2(3) = 1.6/(1-\delta_2)$

For an equilibrium we need:
$\delta_1 8/(1-\delta_1^2) = V_1(1) \geq 8 + \delta_1 V_1(3) = 8 + \delta_1 1.6/(1-\delta_1)$
which implies $\delta_1 8 \geq 8(1-\delta_1^2) + \delta_1 1.6(1+\delta_1)$ [we use here: $(1-\delta_1^2) = (1+\delta_1)(1-\delta_1)$]
and so $6.4\delta_1^2 + 6.4\delta_1 - 8 \geq 0$,
$8/(1-\delta_1^2) = V_1(2) \geq 0 + \delta_1 V_1(3) = 0 + \delta_1 1.6/(1-\delta_1)$
which implies $8 \geq 0(1-\delta_1^2) + \delta_1 1.6(1+\delta_1)$, and so $0 \geq 1.6\delta_1^2 + 1.6\delta_1 - 8$,
$8/(1-\delta_2^2) = V_2(1) \geq 0 + \delta_2 V_2(3) = 0 + \delta_2 1.6/(1-\delta_1)$,
which implies $8 \geq 0(1-\delta_2^2)+\delta_2 1.6(1+\delta_2)$, and so $0 \geq 1.6\delta_2^2 + 1.6\delta_2 - 8$,
$\delta_2 8/(1-\delta_2^2) = V_2(2) \geq 2 + \delta_2 V_2(3) = 2 + \delta_2 1.6/(1-\delta_2)$,
which implies $\delta_2 8 \geq 2(1-\delta_2^2) + \delta_2 1.6(1+\delta_2)$, and so $0.4\delta_2^2 + 6.4\delta_2 - 2 \geq 0$.
Notice, all of these inequalities are satisfied when δ_1 and δ_2 are very close to 1.
The middle two are satisfied for all δ_i between 0 and 1, so only the first and last matter.
The first is satisfied when $\delta_1 \geq (-6.4+(6.4^2+4*6.4*8)^{0.5})/(2*6.4) = 0.7247$.
The last is satisfied when $\delta_2 \geq (-6.4+(6.4^2+4*0.4*2)^{0.5})/(2*0.4) = 0.3066$.