

Econ 20600: Assignment 4, Moral hazard exercises

1. Consider a moral-hazard model where the agent is risk neutral but has limited liability. Suppose the required investment is $K=20$, the project's return is $R=50$ if successful but is 0 if unsuccessful, the probability of success is $p_H=0.6$ if the agent behaves well but is $p_L=0.2$ if the agent misbehaves, the agent's outside option is $w_0=0$, and the agent's private benefit from misbehavior is $B=8$.

- Suppose first that the agent has no collateral ($C=0$) and punishment is not allowed. Find the highest feasible expected net profit such that the agent is willing to participate and behave well.
- Find the optimal contract and optimized net profit with $C=0$ but with punishment allowed.
- How much collateral C would the agent need for our optimal contract without punishment to yield the same expected profit as in part (b) (with punishment but $C=0$).

2. Consider a moral-hazard problem where agents control a Poisson process in which accidents occur at a rate of $\alpha=0.2$ per unit time when the agent's hidden action is diligent a_H , but accidents occur at a rate of $\beta=0.5$ per unit time when the agent hidden action is shirking a_L . Shirking give the agent a hidden income worth $D=6$ per unit time. Agents are risk-neutral, and use the discount rate per unit time $r=0.1$. Agents can get the competitive market wage $w_0=1$, and so the net present-discounted value of an agent's alternative in the labor market is $\bar{u}=w_0/r=10$. Consider an incentive plan in which the agent will be paid a constant wage w until an accident occurs, and then he will be dismissed and return to the competitive labor market forever. What is the lowest wage w that will motivate the agent to always be diligent? What is the expected net present value U_1 of the agent's anticipated compensation in this job?

3. The agent who manages a shop for the principal will choose one of two hidden actions, d_H and d_L . The agent pays a hidden monetary effort cost $c_H = 1000$ if he does d_H , but his effort cost is $c_L = 0$ if he does d_L . The shop's gross profit \tilde{y} (before subtracting the agent's wages) will be a Normal random variable with a given standard deviation $\sigma=3000$, and its mean will be $m_H=9000$ if the agent chooses d_H , but its mean will be $m_L=6800$ if the agent chooses d_L . The agent has a constant risk tolerance $T=1500$, but the principal is risk neutral. The agent's reservation wage is $w_0=5000$, which he can get (with zero effort cost) from other employers.

(a) Suppose that the principal will pay the agent a wage $w(\tilde{y}) = A+B\tilde{y}$ that depends linearly on \tilde{y} , for some constants A and B to be chosen by the principal. What A and B minimize the principal's expected cost of paying the agent, subject to the constraints that he should be willing to manage this shop and choose action d_H ? What is the principal's expected net profit $E(\tilde{y} - w(\tilde{y}))$ under this optimal linear incentive plan?

(b) Compare the principal's expected net profit under the optimal plan that minimizes the principal's expected wage cost subject to the constraints that the agent should be willing to manage this shop and choose the low-effort action d_L . Is it better to motivate d_H or d_L ?

(c) If we changed the risk tolerance, keeping all other parameters the same, what is the lowest risk tolerance T such that the principal prefers to use a linear wage to motivate d_H rather than d_L ?

(d) Consider a two-wage plan where the agent is paid w if $\tilde{y} \geq 3000$ but is paid $w - \Delta$ if $\tilde{y} < 3000$. (Assume $T=1500$ again.) Find the principal's optimal regular-wage w and penalty Δ such that the agent is willing to manage the shop and choose the high effort d_H . Compare the principal's expected wage cost under this plan to the principal's expected wage cost that you found in (b).

[Notes: Utility of $\$x$ is $U(x) = -e^{-x/T}$. When \tilde{y} is Normal with mean μ and standard deviation σ , probabilities can be computed in Excel by $P(\tilde{y} < z) = \text{NORMSDIST}((z - \mu)/\sigma)$ for any number z .]

Econ 20600: Assignment 3, adverse selection and signaling equilibria

1. Suppose that there are 3 bidders to buy an object in an auction, and each bidder i has a private value θ_i for the object that is drawn independently from a uniform distribution on the interval from 0 to 10. The object will be sold to the highest bidder in the auction. The amount that any winner must pay in the auction depends according to some (unspecified) formula on his own bid and all the other bids. Suppose that the bidders are playing according to a symmetric equilibrium, in which the bid of each bidder i depends on his value θ_i according to some strictly increasing function $\beta(\theta_i)$. So the winner of the auction (the high bidder) will be the bidder who actually has the highest true value for the object.

- For any θ_i in $[0,10]$, show the formula for $q_i(\theta_i)$, the conditional probability of bidder i winning the object in the auction given that his value is θ_i .
- Show the conditional expected profit $U(\theta_i)$ for bidder i in the auction when his value is θ_i .
- Show the conditional expected payment $x_i(\theta_i)$ that bidder i will make in the auction given that his value is θ_i (but not knowing whether he will win the auction or not).
- Compute the expected total payment to the seller in this auction (which is 3 times the expected payment from each bidder before his type is specified).
- Now suppose that each bidder i will pay his bid $\beta(\theta_i)$ if he wins but pays nothing otherwise (a first-price auction). What must be the equilibrium bid for a bidder whose value-type is θ_i ?

2. A rancher's cattle are forbidden to trespass on a neighboring farmer's fields unless the farmer sells a right of trespass to the rancher. Permitting the rancher's cattle to trespass in these fields would give the rancher some benefit R , but it would cause some cost T for the farmer. The rancher privately knows her benefit R , but the farmer thinks that R was drawn from a Uniform distribution on $[0,2]$. The farmer privately knows his cost T , but the rancher thinks that T was drawn from a Uniform distribution on $[1,3]$.

A benevolent government official is planning help arbitrate an agreement for the farmer to sell a right of trespass to the rancher if and only if $R \geq T$. If necessary, the government can subsidize or tax the transaction. (That is, the rancher may pay a different amount from what the farmer is paid, the difference being a government subsidy or tax.) But there should be no monetary payments unless the right is actually transferred (which happens in this arbitration only if $R \geq T$). Both parties must be given appropriate incentives to reveal their private information.

- Compute the expected net total social gains $E(\max\{R-T,0\})$ that this arbitration can achieve.
- For each T in $[1,3]$, show the conditional probability $q_1(T)$ that the farmer will grant a right of trespass to the rancher in this arbitration if the farmer's cost-type is T .
- For each T in $[1,3]$, show the conditional expected profit $U_1(T)$ for the farmer in this arbitration if her cost-type is T .
- For each R in $[0,2]$, show the conditional probability $q_2(R)$ that the rancher will get the right of trespass in this arbitration if the rancher's benefit-type is R .
- For each R in $[0,2]$, compute the conditional expected profit $U_2(R)$ for the rancher in this arbitration if his benefit-type is R .
- Compute the expected net subsidy that the government official will have to provide in this arbitration. (Hint: the sum of the expected profits for the rancher and farmer can be greater than the expected total social gains only if they expect a positive net subsidy from the government.)

3. Consider a market where each seller has one widget to sell, but the quality of the widget is unverifiable private information known only by the seller. Depending on its quality, the widget could be worth anywhere from \$10 to \$30 to the seller, and the distribution of these sellers' values is Uniform on this interval. For any qualified buyer, the widget would be worth \$7 more than it is worth to the seller. There are many more qualified buyers than sellers in the market.

- Suppose that all trading of widgets must go through a Walrasian auctioneer who posts a price β for all trades. What price can yield an equilibrium in which a positive fraction of sellers actually trade and there is no rationing of buyers?

- (b) Now suppose that each seller can post a price at which he is thereafter committed to trade if any buyer accepts his offer, and otherwise he must keep his widget. In equilibrium, the seller's probability of trading may depend on his posted price. Find a market equilibrium in which each type seller posts a price that is actually equal to the buyer's true value of the widget, and the probability of trading is 1 if the posted price is \$17 but the probability of trading declines for higher prices. Show a formula for the expected gains from trade for each type of seller. (This is the best safe trading plan.)
- (c) Identify the set of seller-types that get higher expected gains in the separating plan of part (b) than in the pooling plan of part (a).
- (d) Again assume that sellers post prices, as in (b). Find a two-price equilibrium in which some widgets are sold for \$33, there is one other lower price y at which some widgets are sold, there is no excess demand at either of these prices, and no sellers post any other prices besides \$33 and y .
- (e) If a buyer and seller were isolated from this market, so each could trade only with the other, and the buyer could bid any price β as a first-and-final offer, what would be the buyer's optimal price? Show a formula for the buyer's expected gains from trade as a function of her bid β .

4. Each worker's talent is measured by his type t_1 that is independently drawn from a Uniform distribution on the interval from 0 to 1. Each worker knows his own talent-type, but nobody else can observe it. Hiring a worker of type t_1 would have productive value $10t_1$ to any firm. Because of competition among firms, each worker will be offered a wage equal to his conditionally expected productive value, given what is publicly observable about him.

A worker's education does not affect his productivity, but his educational achievement is publicly observable, and the effective cost of getting education is higher for less talented workers.

Suppose that the effective cost of getting d years of education is d/t_1 for a worker of type t_1 , and a worker's objective is to maximize his wage-offer minus the effective cost of his education.

- (a) Find an equilibrium in which each type- t_1 worker gets 0 years of education if $0 \leq t_1 < 0.5$ but gets some positive number D years of education if $0.5 < t_1 \leq 1$. (So the competitive wage offered to worker with 0 years of education is $E(10t_1 | t_1 < 0.5) = 2.5$, and the competitive wage offered to workers with D years of education is $E(10t_1 | t_1 > 0.5) = 7.5$.) What D makes this an equilibrium?
- (b) Find an equilibrium in which workers with types above some number θ get $d=4$ years of education and are offered a high wage H , and all lower-type workers get $d=0$ years of education and are offered a low wage L . What must θ , H , and L be to make this an equilibrium?
- (c) Find an equilibrium in which each type t_1 chooses a separate education $d(t_1)$ and is paid $10t_1$.

5. Consider a bilateral trading problem where agent 1 is the seller of a unique object and 1's type may be either H or L, with probabilities p_H and $p_L = 1 - p_H$ respectively. Agent 2 is the buyer. The object would be worth \$100 to 1 or \$190 to 2 if 1's type is L, but it would be worth \$200 to 1 or \$220 to 2 if 1's type is H.

- (a) Find the best safe incentive-compatible trading plan (best for seller given safety for buyer with any type).
- (b) Show that the separating plan is interim pareto-dominated by a pooling plan when $p_H = 0.9$.
- (c) With $p_H = 0.9$, in a market where sellers can get the pooling plan from (b), show an alternative offer (x, q) that would attract only high sellers and would yield a positive profit for buyers who trade with them.
- (d) Find the lowest value of p_H such that the buyer's optimal trading plan would buy from both types.
- (e) Now suppose that there is a costly public signal that the seller could make before approaching the buyer, and the seller's cost of this signal would be $c_L > 0$ or $c_H > 0$, depending on the his type.

After choosing to make the signal or not, the seller then approaches the buyer and makes a first-and-final offer to sell his object for any selected price, which the buyer can only accept or reject. Consider equilibria where the buyer's belief about the seller will depend on the seller's signaling decision but not on the offer.

[e1] What conditions must c_L and c_H satisfy for there to be such an equilibrium where the seller would choose to signal when his type is H but not when his type is L?

[e2] What conditions must c_L and c_H satisfy for there to be such an equilibrium where both types of the seller would choose to signal? (These conditions may depend on the parameter p_H .)

Econ 20600: Assignment 2, adverse-selection exercises

1. We are to design an optimal contract for a principal who needs to buy the services of a skilled artisan when the artisan's cost of supplying his services are not known by the principal.

If the artisan's cost type is t , then his net utility gain from providing q hours of service and receiving wage x would be $u(x, q | t) = x - tq$. The principal's net benefit from paying x for q hours of service from either type would be $6\sqrt{q} - x$. The number of hours worked (which the principal can observe) and the wage can depend on the artisan's cost type (which the artisan knows), but the artisan could misrepresent his type. The artisan also will not work unless he gets a nonnegative utility gain.

(a) Suppose that the artisan's cost type is either 3 or 5, and the principal thinks that the probabilities are $P(t=3) = 2/3$ and $P(t=5) = 1/3$. Find the contract that maximizes the principal's expected net benefit from her business with the artisan. Show both the amount of service $q(t)$ and the wage payment $x(t)$ for each possible type t .

(b) If we changed part (a) by assuming that the principal would get $A_H\sqrt{q} - x$ from paying x for q hours of service from a high-type artisan with cost 5, for some parameter A_H (keeping $A_L=6$), what the bound on A_H defines the range of parameters such that the binding constraints remain the same for the principal's optimal contract?

Compute the optimal contract for A_H being this bound plus 1.

(c) Now suppose instead that the artisan's cost type is drawn from a Uniform distribution on the interval from 3 to 5. As in part (a), the principal's net benefit from paying x for q hours of service from any type would be $6\sqrt{q} - x$.

Find the contract that maximizes the principal's expected net benefit from her business with the artisan. Show formulas for computing the amount of service $q(t)$ and the wage payment $x(t)$ for each possible type t .

Compute the exact numerical values of these formulas at $t=3$ and $t=5$; that is, compute $q(3)$, $x(3)$, $q(5)$, and $x(5)$.

2. A translator's cost per page is \$5/page if he is the fast type and is \$10/page if he is the slow type. Suppose that the principal's monetary benefit from getting q pages translated is $S(q) = (220q - q^2)/12$ for any $q \leq 110$.

(a) If the principal can observe the translator's type, what is the optimal plan for deciding how many pages he should translate and how much he should be paid, depending on his type?

(b) Now suppose that the principal cannot observe the translator's type. The translator knows his type but can misrepresent it. What would be the result of offering the plan from part (a)?

(c) With the principal unable to observe the translator's type, find the optimal contract that the principal should offer, to maximize her expected net profit (benefit minus wage-cost), when the principal believes that the probability of the translator being slow is $P(\text{slow}) = 1/2$.

(d) How would your answer to part (c) change if $P(\text{slow}) = 1/4$?

(e) For what range of values of $P(\text{slow})$ would it be optimal for the principal to offer a contract such that the slow-type translator would do no work at all (0 pages for \$0 wages)?

(f) Now suppose instead that the translator's cost is drawn from a Uniform distribution on the interval from \$5 to \$10 per page. The translator knows his type but can misrepresent it. Find the incentive-compatible plan that maximizes the principal's expected net profit.

Econ 20600: Assignment 1, Exercises on Risk Aversion and Conditional Expectation

1. Suppose that a decision-maker has a constant risk tolerance of \$6000. For each of the following lotteries, compute the decision-maker's certainty equivalent and risk premium.
 - (a) A lottery that pays either \$2000 or \$6000 or \$10,000, each with probability 1/3.
 - (b) A lottery that pays \$2000 or \$6000 or \$10,000, with probabilities 0.4, 0.2, 0.4 respectively.
 - (c) A lottery that pays \$0 or \$4000 or \$8000, with probabilities 0.4, 0.2, 0.4 respectively.
 - (d) A lottery that pays \$0 or \$8000 or \$16,000, with probabilities 0.4, 0.2, 0.4 respectively.
2. Redo problem 1, computing the certainty equivalent and risk premium for each of the four lotteries in (a)-(d), but now suppose that the decision-maker has utility function $u(x) = x^{0.5}$ (his utility equals the square root of his monetary payoff from the lottery).
3. Consider a risky investment that pays net earnings drawn from a Normal distribution with mean \$9000 and standard deviation \$3000. Investor 1 thinks that this investment would have a certainty-equivalent value \$6000. Investor 2 thinks that this investment would have a certainty-equivalent value \$5000. Suppose that both investors have constant risk tolerance.
 - (a) What is investor 1's risk tolerance?
 - (b) What is investor 2's risk tolerance?
 - (c) Now suppose that investor 1 actually has an investment that will pay net earning drawn from a Normal distribution with mean \$9000 and standard deviation \$3000, but he can offer to sell part of this investment to individual 2. To maximize the sum of their certainty equivalents, what share of this investment should investor 1 offer to sell to investor 2? What is the highest price that investor 2 would be willing to pay now to buy this optimal share of the investment?
4. Individuals 1 and 2 are a brother and sister who have inherited equal shares of their mother's scattered real estate properties. They have constant risk tolerances $T_1 = \$20,000$ and $T_2 = \$45,000$ respectively. Appraisals of the property by independent real-estate experts suggest that the total returns from selling the properties individually over a period of months may be viewed as a draw from a Normal distribution with mean \$200,000 and standard deviation \$75,000. But one real estate agent just offered to buy all the properties right now for \$150,000, provided that they accept the offer within 24 hours.
 - (a) Show that, if they were constrained to share equally all income from the sale of the properties, then 1 and 2 would disagree gamble was preferable to the certain \$150,000.
 - (b) Suppose that, because of their disagreement, the real-estate agent's 24-hour offer has been lost, and so their prospective returns now are defined by the Normal($\mu = \$200,000, \sigma = \$75,000$) distribution. Suppose 2 can make an offer acquire any part of her brother's 50% share, for any price now that would be acceptable to her brother. What share should 2 try to acquire, and what is the smallest payment that 1 would accept now in exchange for this reduction of his share of the risks? What is the maximum amount that 2 would be willing to pay to acquire this increased share (above the 50% that she already owns)?
 - (c) Show that, after their shares have been adjusted as in part (b), if the real-estate agent extended the \$150,000 offer for another 24 hours, both 1 and 2 would want to turn it down. What is the smallest price for the entire estate that they would be willing to accept?

5. A construction project has two stages. Let X denote the cost of the first stage and let Y denote the cost of the second stage (both in thousands of dollars). Suppose that the joint distribution of these costs is as follows:

| | $Y = 200$ | $Y = 250$ | $Y = 300$ |
|----------|-----------|-----------|-----------|
| $X = 90$ | 0 | .15 | .15 |
| $X = 70$ | .10 | .20 | .16 |
| $X = 50$ | .09 | .09 | .06 |

(a) Compute the expected value of Y . (You should find that $E(Y) > 257$, and so a risk-neutral builder would prefer to pay 257 than Y for the second stage of the construction project.)

(b) Compute the conditionally expected value of Y given each of the possible values of X , and show that the formula $E(Y) = \sum_m P(X=m) * E(Y | X=m)$ is satisfied for this example.

(c) The builder who will pay these costs X and Y is risk neutral. But suppose now that he has an option to outsource the second stage of the project to another contractor for the price 257.

The builder does not know X yet. What is the maximum that the builder should be willing to pay to learn X before he has to decide about whether to pay the other contractor 257 to do the second stage for him?

(d) Suppose that the builder must decide about whether to subcontract the second stage for 257 now, before learning X ; but there is a higher price Q at which he could subcontract the second stage after he finishes the first stage and learns X . What is the highest value of Q for which the builder should reject the current offer of 257?

6. A and B are unknown quantities. Given my current information only, I think that $E(A) = 3000$. But if I knew the value of B , I would revise my beliefs about A to $E(A|B=0) = 1000$ or $E(A|B=1) = 9000$. This quantity B can only equal 0 or 1.

(a) What is $P(B=0)$?

(b) Suppose that I am a risk-neutral investor, and I have an option to buy an asset that will be worth A next month, for which I must pay the price \$5000 now. What is the maximum that I should be willing to pay to learn the value of B before deciding whether to pay \$5000 for A ?

(c) How would your answer to (b) change if the price to buy A now were \$2000 instead of \$5000?