CONDITIONAL EXPECTATION

For any event A and any random variable **Y**, the conditionally expected value of **Y** given A, denoted E(**Y**|A), is the expected value of **Y** that you would assess after you learned that the event A occurred. That is, when **Y** is a discrete random variable,

$$E(\mathbf{Y}|A) = \sum_k P(\mathbf{Y}=k|\ A) * k,$$

where the summation is over all k that are possible values of **Y**. The event A on which we condition may itself be the observation of a particular value of another random variable.

For example, suppose we anticipate that the value of the dollar in yen will change next week depending on several forthcoming government announcements. The first of these announcements is an estimate of the quarterly increase in the US money supply, which will be announced on Thursday morning. Other important trade deficit statistics will be announced Friday morning. Let **Y** denote the value of the dollar in yen that will prevail on Friday afternoon, after the deficit announcement, and let **M** denote the money-supply increase that will be announced on Thursday. For simplicity, let us assume (albeit unrealistically) that the major currency traders now agree on the following joint distribution for these two random variables:

	Y=126	**Y**=128	**Y**=130	**Y**=132	(Row total)
M=.02	.10	.15	.05	0	(.30)
M=.01	.05	.15	.15	.05	(.40)
M=0	0	.10	.10	.10	(.30)
(Total)	(.15)	(.40)	(.30)	(.15)	(1.0)

The expected value of **Y** (given only our current information) is

$$E(\mathbf{Y}) = .15*126 + .40*128 + .30*130 + .15*132 = 128.9.$$

Let us next ask what may be the conditionally expected value of **Y** after the money-supply increase is revealed. The conditionally expected value of **Y** given a two-percent increase in the money supply would be

$$E(\mathbf{Y}|\ \mathbf{M}=.02) = .333*126 + .500*128 + .167*130 + 0 * 132 = 127.67.$$

because .10/30 = .333, .15/.30 = .500, .05/.30 = .167, and 0/.30 = 0.

On the other hand, the conditionally expected value of **Y** given a one-percent increase, or a zero-percent increase, would be

$$E(\mathbf{Y}|\ \mathbf{M}=.01) = .125*126 + .375*128 + .375*130 + .125*132 = 129,$$

$$E(\mathbf{Y}|\ \mathbf{M}=0) = 0 * 126 + .333*128 + .333*130 + .333*132 = 130.$$

The following general formula holds for any two random variables **Y** and **M**:

$$E(\mathbf{Y}) = \sum_m P(\mathbf{M}=m) * E(\mathbf{Y}|\ \mathbf{M}=m).$$

(Here the summation is over all m that are possible values of **M**.) This formula asserts that, before we observe **M**, our expected value of **Y** is equal to the expected value of what we will think is the expected value of **Y** after we observe **M**. Or, we may briefly (but obscurely) say "the expected value of the expected value is equal to the expected value." This formula underlies the analysis of decision trees. In the above example, this formula asserts that

$$E(\mathbf{Y}) = P(\mathbf{M}=0) * E(\mathbf{Y}|\mathbf{M}=0) + P(\mathbf{M}=.01) * E(\mathbf{Y}|\mathbf{M}=.01) + P(\mathbf{M}=.02) * E(\mathbf{Y}|\mathbf{M}=.02)$$
$$= .30 * 130 + .40 * 129 + .30 * 127.67 = 128.9$$

This general formula can be proven as follows (using the basic formula for conditional probabilities at the third step):

$$\sum_m P(\mathbf{M}=m) * E(\mathbf{Y}|\mathbf{M}=m) = \sum_m P(\mathbf{M}=m) * \sum_k P(\mathbf{Y}=k|\mathbf{M}=m) * k$$
$$= \sum_k \sum_m P(\mathbf{M}=m) * P(\mathbf{Y}=k|\mathbf{M}=m) * k$$
$$= \sum_k \sum_m P(\mathbf{M}=m \cap \mathbf{Y}=k) * k$$
$$= \sum_k P(\mathbf{Y}=k) * k = E(\mathbf{Y}).$$

To interpret this result, consider again our exchange-rate example, where **Y** denotes the yen-value of the dollar on Friday afternoon. Ignoring short-term interest considerations, we may suppose that prices in the currency market should be such that a trader would get zero expected profits from buying yen on one day and converting the yen back to dollars on another day. (Otherwise, risk-neutral traders would want to make infinite currency transactions.) This condition implies that the yen-value of the dollar now (or on Wednesday), before **M** is announced, should be E(**Y**), so that a trader would get zero expected profit from buying yen now and converting them back to dollars on Friday. Similarly, for any number m, if it is announced on Thursday morning that the money-supply increase (**M**) is equal to m, then the yen-value of the dollar on Thursday afternoon should be equal to $E(\mathbf{Y}|\ \mathbf{M}=m)$, so that a trader would get zero expected profits from buying yen after the money-supply announcement and converting them back to dollars on Friday afternoon. Then the formula

$$E(\mathbf{Y}) = \sum_m P(\mathbf{M}=m) * E(\mathbf{Y}|\ \mathbf{M}=m)$$

tells us that a trader would also get zero expected profits from buying yen now and converting them back to dollars after the money-supply announcement on Thursday afternoon.

As another application of this formula, let A denote any event, and define **X** to be a random variable such that **X** = 1 if the event A is true, but **X** = 0 if the event A is false. Then

$$E(\mathbf{X}) = P(A) * 1 + (1 - P(A)) * 0 = P(A).$$

That is, the expected value of this random variable **X** is equal to the probability of the event A. Now suppose that **W** is any other random variable such that observing the value of **W** might cause us to revise our beliefs about the probability of A. Then the conditionally expected value of **X** given the value of **W** would be similarly equal to the conditional probability of A given the value of **W**. That is, for any number w that is a possible value of the random variable **W**,

$$E(\mathbf{X}|\ \mathbf{W}{=}w) = P(A|\ \mathbf{W}{=}w).$$

Thus, the general equation

$$E(\mathbf{X}) = \sum_w P(\mathbf{W}{=}w) * E(\mathbf{X}|\ \mathbf{W}{=}w).$$

gives us the following probability equation:

$$P(A) = \sum_w P(\mathbf{W}{=}w) * P(A|\ \mathbf{W}{=}w).$$

This equation says that the probability of A, as we assess it given our current information, must equal the the current expected value of what we would think is probability of A after learning the value of **W**. Learning **W** might cause us to revise our assessment of the probability of A upwards or downwards, but the weighted average of these possible revisions, weighted by their likelihoods, must be equal to our currently assessed probability P(A).

In our example with **Y** and **M** above, consider the event that **Y**>129. From the marginal distribution of **Y** we see that

$$P(\mathbf{Y}{>}129) = .30{+}.15 = .45.$$

But when we learn the value of **M**, the conditional probability of this event given **M** could be

$$P(\mathbf{Y}{>}129|\mathbf{M}{=}.02) = .167{+}0 = .167,$$

$$P(\mathbf{Y}{>}129|\mathbf{M}{=}.01) = .375{+}.125 = .500,$$

$$P(\mathbf{Y}{>}129|\mathbf{M}{=}.02) = .333{+}.333 = .667.$$

The expected value of these posterior conditional probabilities is indeed equal to the current probability of this event, because

$$\sum_m P(\mathbf{M}{=}m) * P(\mathbf{Y}{>}120|\mathbf{M}{=}m) = .30*.167{+}.40*.500{+}.30*.667 = .450 = P(\mathbf{Y}{>}129).$$

Now let us consider the how to measure the expected value of information for use in a decision problem. In our basic example above, suppose that a risk-neutral individual has an option now to buy \$1 for 129 yen. Suppose also that he could pay some extra amount x (in yen) for the right to wait and exercise this option after **M** is revealed. What is the highest amount x such that this individual should be willing to pay x to get the right to observe **M** before deciding about whether to exercise this option or not?

Assuming that the market is dominated by investors who are risk neutral and share the beliefs that we described above (and who do not discount significantly between now and next friday), the current value of a dollar today should be $E(\mathbf{Y}) = 128.9$, and so such an option would have value 0 now. After learning **M**, the investor would still think that the option was worth nothing if M= .02 or if **M**=.01, but if **M**= 0 then the option will be worth 130-129=1 yen to this risk-neutral investor. Thus, as the latter event has probability .30, the information would increase the value of the option to $.30*1 = .3$.

More generally, an option now to buy some an asset that will have future liquidation value **Y** for a different striking price S should be worth

$$\max\{E(\mathbf{Y})-S, 0\}$$

to a risk-neutral investor, if he must exercise this option now or never. But if he could exercise this option after learning **M** then his expected value of the option (assessed now, before **M** is learned) would become

$$\sum_m P(\mathbf{M}=m) * \max\{E(\mathbf{Y}|\mathbf{M}=m)-S, 0\}$$

The difference

$$\sum_m P(\mathbf{M}=m) * \max\{E(\mathbf{Y}|\mathbf{M}=m)-S, 0\} - \max\{E(\mathbf{Y})-S, 0\}$$

is the expected value of information about **M** for an individual who has to decide about whether to exercise this option. For example, if the striking price of the option were S=128, then the expected value of information about **M** would be (in yen):

$$[.30*0 + .40*(129-128) + .30*(130-128)] - (128.9-128) = 1 - .9 = .1$$