First adverse-selection problem: one privately-informed agent with two possible types

The agent could be one of two types, θ_H or θ_L where $\theta_H > \theta_L$. His type is his private information.
The principal thinks the probability of θ_H is p_H, and the probability of θ_L is $p_L = 1 - p_H$.
The agent's effort q and wage x are both observable numbers which may depend on the agent's reported type, but the agent can misrepresent his type. Effort must be nonnegative $q \ge 0$.
When the agent's type is θ, any wage x and effort $q \ge 0$ yield payoff $x - \theta q$ for the agent and yield profit $S(q|\theta) - x$ for the principal. The agent's alternative pays $w_0 = 0$ for effort 0.
Suppose $S(0|\theta)=0$, $S'(q|\theta)>0$, $S''(q|\theta) \le 0$, $\forall\theta, \forall q$, and $S'(0|\theta_L) > \theta_L$. [$'$ for $\partial/\partial q$]

The principal's problem is to choose a contract-menu (q_H, x_H, q_L, x_L) to maximize $p_H(S(q_H|\theta_H) - x_H) + p_L(S(q_L|\theta_L) - x_L)$ subject to $x_L \in \mathbb{R}$, $x_H \in \mathbb{R}$, $q_L \ge 0$, $q_H \ge 0$,

$U_L = x_L - \theta_L q_L \ge w_0$, [L-participation, λ_L]
$U_H = x_H - \theta_H q_H \ge w_0$, [H-participation, λ_H]
$x_L - \theta_L q_L \ge x_H - \theta_L q_H$, [H|L-incentive, $\alpha_{H|L}$]
$x_H - \theta_H q_H \ge x_L - \theta_H q_L$. [L|H-incentive, $\alpha_{L|H}$]

The Lagrangean can be written: $\mathcal{L}(q,x;\lambda,\alpha) =$

$$= p_H(S(q_H|\theta_H) - x_H) + p_L(S(q_L|\theta_L) - x_L) + \lambda_L[x_L - \theta_L q_L - w_0] + \lambda_H[x_H - \theta_H q_H - w_0]$$
$$+ \alpha_{H|L}[x_L - \theta_L q_L - x_H + \theta_L q_H] + \alpha_{L|H}[x_H - \theta_H q_H - x_L + \theta_H q_L]$$
$$= p_H S(q_H|\theta_H) - q_H[(\lambda_H + \alpha_{L|H})\theta_H - \alpha_{H|L}\theta_L] + p_L S(q_L|\theta_L) - q_L[(\lambda_L + \alpha_{H|L})\theta_L - \alpha_{L|H}\theta_H]$$
$$+ x_H[-p_H + \lambda_H + \alpha_{L|H} - \alpha_{H|L}] + x_L[-p_L + \lambda_L + \alpha_{H|L} - \alpha_{L|H}].$$

The first-order Lagrange optimality conditions for $x_L \in \mathbb{R}$ and $x_H \in \mathbb{R}$ are

$0 = \partial\mathcal{L}/\partial x_L = -p_L + \lambda_L + \alpha_{H|L} - \alpha_{L|H}$, and so $\lambda_L + \alpha_{H|L} = p_L + \alpha_{L|H}$;
$0 = \partial\mathcal{L}/\partial x_H = -p_H + \lambda_H + \alpha_{L|H} - \alpha_{H|L}$, and so $\lambda_H + \alpha_{L|H} = p_H + \alpha_{H|L}$.

With these equation, the Lagrangean simplifies to

$$\mathcal{L} = p_H\{S(q_H|\theta_H) - q_H[\theta_H + (\theta_H - \theta_L)\alpha_{H|L}/p_H]\} + p_L\{S(q_L|\theta_L) - q_L[\theta_L + (\theta_L - \theta_H)\alpha_{L|H}/p_L]\}.$$

With $\theta_L < \theta_H$ and $q_H \ge 0$, the L-participation constraint is implied by the H-participation and H|L-incentive constraints. So L-participation is a redundant constraint; its multiplier is $\lambda_L = 0$.
The incentive constraints (L|H first, then H|L) imply: $\theta_H(q_L - q_H) \ge x_L - x_H \ge \theta_L(q_L - q_H)$.
With $\theta_H > \theta_L$, this implies $q_L - q_H \ge 0$ and $x_L - x_H \ge 0$, so $q_L \ge q_H$ and $x_L \ge x_H$.
[If $q_L > q_H$ then: $x_H - t\,q_H \ge x_L - t\,q_L \iff t \ge (x_L - x_H)/(q_L - q_H)$. So any cost type $t > \theta_H$ would prefer (q_H, x_H) over (q_L, x_L), while any cost type $t < \theta_L$ would prefer (q_L, x_L).]

Suppose (q_H, q_L) maximize the Lagrangean. $S'(0|\theta_L) > \theta_L \ge \theta_L + (\theta_L - \theta_H)\alpha_{L|H}/p_L$, so $q_L > 0$.
With $\lambda_L = 0$, satisfying $0 = \partial\mathcal{L}/\partial x_L$ requires $\alpha_{H|L} > 0$, so the H|L-incentive constraint is binding.
So we have $\alpha_{H|L} - \alpha_{L|H} = p_L$ and $\lambda_H = p_H + \alpha_{H|L} - \alpha_{L|H} = 1$. So $U_H = w_0$ and $U_L = x_H - \theta_L q_H$.
Now we have two cases (separating and pooling):

In <u>case 1</u>, we have $\alpha_{L|H} = 0$. Then we get $\alpha_{H|L} = p_L$, $\lambda_H = 1$, $S'(q_L|\theta_L) = \theta_L$,
$S'(q_H|\theta_H) \le \theta_H + (\theta_H - \theta_L)p_L/p_H$ and $q_H \ge 0$ with at least one equality (complementary slackness),
$U_H = w_0$, $x_H = w_0 + \theta_H q_H$, $U_L = x_H - \theta_L q_H$, and $x_L = \theta_L q_L + U_L$.
If these conditions yield $q_H \le q_L$, then the $\alpha_{L|H}$-constraint is satisfied and this is the solution.

Otherwise, we have <u>case 2</u> where $\alpha_{L|H} > 0$. So both incentive constraints bind, so $q_H = q_L = q^* > 0$.
The pooling q^* satisfies $S'(q^*|\theta_H) = \theta_H + (\theta_H - \theta_L)\alpha_{H|L}/p_H$, $S'(q^*|\theta_L) = \theta_L - (\theta_H - \theta_L)\alpha_{L|H}/p_L$.
So q^* can be computed from $p_H S'(q^*|\theta_H) + p_L S'(q^*|\theta_L) = \theta_H$, because $\alpha_{H|L} - \alpha_{L|H} = p_L = 1 - p_H$.
Then $\alpha_{L|H} = p_L[\theta_L - S'(q^*|\theta_L)]/(\theta_H - \theta_L) > 0$ requires $S'(q^*|\theta_L) < \theta_L$ for this pooling plan.

Motivating an agent with a type drawn from a continuous distribution.

Suppose that the agent's type $\tilde{t}$ is a random variable drawn from an interval $[A,B]$.

The agent's type is his cost of effort, and his utility for income x and effort q is $x - t\,q$.

Consider any contract $(x(\bullet),q(\bullet))$ where the terms of trade for each type t would be $(x(t),q(t))$.

Let $U(x,q|t) = x(t) - t\,q(t)$ denote the expected utility of type t under this contract.

For any pair of possible types t and s in $[A,B]$, the $(s|t)$-informational incentive constraint says

$U(x,q|t) = x(t) - t\,q(t) \geq x(s) - t\,q(s) = U(x,q|s) + (s-t)q(s)$.

Similarly, the $(t|s)$-incentive constraint implies $U(x,q|s) \geq U(x,q|t) + (t-s)q(t)$

So the $(t|s)$ and $(s|t)$ constraints together imply $(s-t)q(t) \geq U(x,q|t)-U(x,q|s) \geq (s-t)q(s)$.

So when $s > t$ we must have $q(t) \geq q(s)$, and so $q(t)$ is a decreasing function of the cost-type t.

Applying these inequalities over many small steps from t up to B, we get the

<u>informational-rent</u> equation: $U(x,q|t) = U(x,q|B) + \int_t^B q(s)\,ds = U(x,q|B) + \int_0^{q(t)} [q^{-1}(\gamma)-t]\,d\gamma$.

The expected income of type t is then $x(t) = U(x,q|t) + tq(t) = U(x,q|B) + \int_t^B q(s)\,ds + tq(t)$.

Suppose the principal's beliefs about the agent's type are described by the cumulative distribution $F(t) = P(\tilde{t} \leq t)$, and $f(t) = F'(t)$ is the continuous probability density of this distribution, with $f(t)>0$ for all t in $[A,B]$. Here $F(B)=1$, $F(A)=0$, and $P(a\leq\tilde{t}\leq b) = F(b)-F(a)=\int_a^b f(t)\,dt$ whenever $a \leq b$.

Then the expected wage bill $E(x(\tilde{t}))$ is

$$\int_A^B x(t)\,f(t)\,dt = \int_A^B [U(x,q|B) + \int_t^B q(s)\,ds + t\,q(t)]f(t)\,dt$$

$$= U(x,q|B) + \int_A^B \int_t^B q(s)\,ds\,f(t)\,dt + \int_A^B t\,q(t)\,f(t)\,dt$$

$$= U(x,q|B) + \int_A^B \int_A^s f(t)\,dt\,q(s)\,ds + \int_A^B s\,q(s)\,f(s)\,ds$$

$$= U(x,q|B) + \int_A^B F(s)\,q(s)\,ds + \int_A^B q(s)\,s\,f(s)\,ds = U(x,q|B) + \int_A^B q(s)\,[F(s) + s\,f(s)]\,ds$$

$$= U(x,q|B) + \int_A^B q(t)\,[t + F(t)/f(t)]\,f(t)\,dt\,.$$

So the incentive-compatible expected wage $E(x(\tilde{t}))$ looks like what the principal would have to pay without incentive constraints if the cost of each type t were increased to a <u>virtual cost</u> $t+F(t)/f(t)$.

This virtual-cost formula expresses the fact that, when we ask more effort from any type t, we increase the amount that we must pay all types below t, because of incentive constraints.

Suppose paying $\hat{x}$ for effort $\hat{q}$ from type t makes the principal's net gain equal to $S(\hat{q},t)-\hat{x}$.

The optimal trading plan for the principal should maximize $E[S(q(\tilde{t}),\tilde{t})-x(\tilde{t})]$.

When the plan $(x(.),q(.))$ satisfies the incentive constraints, this expected gain is

$$\int_A^B [S(q(t),t) - x(t)]\,f(t)\,dt = \int_A^B [S(q(t),t) - q(t)\,(t+F(t)/f(t))]\,f(t)\,dt - U(x,q|B).$$

To maximize this integral, for each t, choose $q(t)\geq 0$ to maximize $S(q(t),t)-q(t)[t+F(t)/f(t)]$.

If this $q(t)$ is decreasing in t, then it satisfies the incentive constraints with $x(t) =\int_t^B q(s)ds+tq(t)$.

With $x(B) = Bq(B)$, we get $U(x,q|B)=0$, and all participation constraints are satisfied.

Example: Akerlof's Lemons. Suppose the "agent" is the seller of a unique object, of which the "principal" is the only potential buyer. The seller's type is the value of the object to him, which depends on his unverifiable private information about its quality. Then q(t) can be reinterpreted as the probability of his selling the good if he acts like type t, which must satisfy $0 \leq q(t) \leq 1$, and x(t) is his expected revenue from selling if he acts like type t.

Suppose $\tilde{t}$ is drawn from a Uniform distribution on the interval from 0 to 100, but the value of the object to the buyer also depends on the quality (which the buyer would learn only after the transaction) and would be $1.5\tilde{t}$. That is, the object would always be worth 50% more to the buyer.

If (x,q) satisfies the incentive constraints and $U(x,q|t) \geq 0$, the buyer's expected gain from trade is

$\int_0^{100} [1.5tq(t)-x(t)]f(t)dt = \int_0^{100} [1.5t-t-F(t)/f(t)]q(t)f(t)dt-U(x,q|100)$

$= \int_0^{100} [1.5t-2t]q(t)dt/100 - U(x,q|100) \leq 0$. The buyer can only expect to lose if any $q(t)>0$.

Facts about Uniform distributions. Suppose that $\tilde{X}$ is a random variable drawn from a Uniform distribution on the interval from A to B, for some given numbers A and B such that $A < B$.

Then $E(\tilde{X}) = (A+B)/2$. Furthermore, for any number θ between A and B:

$F(\theta) = P(\tilde{X} \leq \theta) = P(\tilde{X} < \theta) = (\theta-A)/(B-A)$, $f(\theta) = F'(\theta) = 1/(B-A)$, $F(\theta)/f(\theta) = \theta - A$,

$E(\tilde{X}|\tilde{X} \leq \theta) = E(\tilde{X}|\tilde{X} < \theta) = (A+\theta)/2$, $E(\tilde{X}|\tilde{X} \geq \theta) = E(\tilde{X}|\tilde{X} > \theta) = (\theta+B)/2$.

Let's do the analogous result for the case where **the agent is a buyer** of some object.

Here the agent's type t is interpreted as his valuation of the object.

Now let q(t) denote the probability of the agent buying the object if her type is t,

and let x(t) denote expected amount that the agent will have to pay if her type is t.

(If w(t) denotes the price that the type-t agent will pay if she buys, and if she would pay nothing if she does not buy the object, then our x(t) is equal to q(t) w(t).)

So the expected gains from trade for a type-t buyer are $U(x,q|t) = t\, q(t) - x(t)$.

An incentive compatible trading plan must satisfy, for all types s and t in the interval [A,B],

$U(x,q|t) = t\, q(t) - x(t) \geq \max_{s\in[A,B]} (t\, q(s) - x(s)) = \max_{s\in[A,B]} U(x,q|s) + (t-s)q(s)$.

With differentiability, $0 = \partial/\partial s\, [t\, q(s) - x(s)]|_{s=t} = t\, q'(t) - x'(t)$ implies

the envelope theorem $U'(x,q|t) = q(t)$, and so each type t gets the information rent

$U(x,q|t) = U(x,q|A) + \int_A^t q(s)\, ds = U(x,q|A) + \int_0^{q(t)} [t-q^{-1}(\gamma)]\, d\gamma$.

(Also, $\partial/\partial s\, [t\, q(s) - x(s)] = (t-s)\, q'(s) < 0$ if $s>t$, or >0 if $s<t$, when $q'(s) > 0$.)

So the expected payment from type t is $x(t) = tq(t) - U(x,q|t) = t\, q(t) - \int_A^t q(s)\, ds - U(x,q|A)$.

The overall expected payment from the buyer, before her type is not known, is

$\int_A^B x(t)\, f(t)\, dt = \int_A^B [t\, q(t) - \int_A^t q(s)\, ds - U(x,q|A)]f(t)\, dt$

$= \int_A^B t\, q(t)\, f(t)\, dt - \int_A^B \int_A^t q(s)\, ds\;\, f(t)\, dt - U(x,q|A)$

$= \int_A^B s\, q(s)\, f(s)\, ds - \int_A^B \int_s^B f(t)\, dt\; q(s)\, ds - U(x,q|A)$

$= \int_A^B [s\, f(s) - (1-F(s))]\, q(s)\, ds - U(x,q|A) = \int_A^B q(t)\, [t - (1-F(t))/f(t)]\, f(t)\, dt - U(x,q|A)$.

Now consider a trading problem where both the seller and buyer have independent private values for the object that they may trade. Let 1 denote the seller, and let 2 denote the buyer.
The seller's type t_1 is what the object is worth to him, and the buyer's type t_2 is what the object would be worth to her. Suppose that each trader's type is his or her private information, but each thinks the other's type is drawn from a cumulative distribution F on the interval [A,B]. $f = F'$.
Now we just put subscripts 1 and 2 on all our past analysis. Let $q_1(t_1)$ and $x_1(t_1)$ denote the conditional probability of trade and the conditional expected payment if the seller's type is t_1.
Let $q_2(t_2)$ and $x_2(t_2)$ denote the conditional probability of trade and the conditional expected payment if the buyer's type is t_2. When we don't know either type, the expected payment is
$\int_A^B x_1(t_1)\, f(t_1)\, dt_1 = \int_A^B x_2(t_2)\, f(t_2)\, dt_2$.
With incentive compatibility for seller and buyer, this equation becomes:

$$U_1(x,q|B) + \int_A^B q_1(t_1)\, [t_1 + F(t_1)/f(t_1)]\, f(t_1)\, dt$$
$$= \int_A^B q_2(t_2)\, [t_2 - (1-F(t_2))/f(t_2)]\, f(t_2)\, dt_2 - U_2(x,q|A).$$

Here $U_1(x,q|B)$ is the expected gains from trade for the highest type of seller, and $U_2(x,q|A)$ is the expected gains from trade for the lowest type of buyer, which are the types least eager to trade.
Let $Q(t_1,t_2)$ denote the probability of trade occuring when the seller is t_1 and the buyer is t_2.
So $q_1(t_1) = \int_A^B Q(t_1,t_2)\, f(t_2)\, dt_2$ and $q_2(t_2) = \int_A^B Q(t_1,t_2)\, f(t_1)\, dt_1$.
Then with participation constraints for the two least-eager-to-trade types, we get

$$0 \le U_1(x,q|B) + U_2(x,q|A)$$
$$= \int_A^B \int_A^B Q(t_1,t_2)\, \{[t_2 - (1-F(t_2))/f(t_2)] - [t_1 + F(t_1)/f(t_1)]\}\, f(t_1)\, f(t_2)\, dt_1\, dt_2.$$

Consider the case of Uniformly distributed types on [0,1]. Then A=0, B=1, $F(t) = t$, $f(t) = 1$, and so we get $0 \le \int_A^B \int_A^B Q(t_1,t_2)\, \{[2t_2 - 1] - [2t_1])\, dt_1\, dt_2 = 2\, E\{Q(\tilde{t}_1,\tilde{t}_2)[\tilde{t}_2 - \tilde{t}_1 - 1/2]\}$.
Thus, the total gains from trade $\tilde{t}_2 - \tilde{t}_1$ must have a conditional expected value of at least 1/2 when trade occurs. But $E(\tilde{t}_2 - \tilde{t}_1 | \tilde{t}_2 > \tilde{t}_1) = [\int_0^1 \int_0^t (t-s)\, 2\, ds\, dt] = 1/3$.

Examples Suppose the agent's has 2 possible cost-types: θ_L=1 or θ_H=2, each with probability 1/2.
The principal's value of effort q from an agent of type θ is $S(q|\theta) = (1+\theta)q^{0.5}$, so the principal's gain from trade when paying x for effort q is $2\sqrt{q} - x$ if $\tilde{\theta}=\theta_L$, $3\sqrt{q} - x$ if $\tilde{\theta}=\theta_H$.
We can apply case 1 from our previous analysis of adverse-selection problem with two types.
The Lagrange multiplier of the H|L incentive constraint is $\alpha = p_L = 0.5$, and the Lagrangean is:
$\mathcal{L} = 0.5\{3q_H^{0.5} - q_H\,[2+(2-1)0.5/0.5]\} + 0.5\{2q_L^{0.5} - q_L[1]\}$. To maximize this over (q_H,q_L), we need $0 = 3(0.5)q_H^{-0.5} - 3$ and $0 = 2(0.5)q_L^{-0.5} - 1$, and so $q_H = 0.25$ and $q_L = 1$.
Because $q_H < q_L$, we know that the case 1 assumptions about binding constraints will be OK here.
The values of w_H and w_L are determined by the binding constraint equations:
To make H-participation binding, $x_H = \theta_H q_H = 2\times0.25 = 0.5$.
To make H|L-incentive binding, $x_L = \theta_L q_L + w_H - \theta_L q_H = 1\times1 + 0.5 - 1\times0.25 = 1.25$.
[If we had changed $S(q_H|\theta_H)$ to be $A_H\sqrt{q_H}$ with some $A_H > 6$, then the above analysis would have yielded $q_H > q_L$, and so the pooling case 2 would have applied.]

A continuous example: Now suppose that $\tilde{\theta}$ is drawn from a Uniform distribution on [1.2], keeping everything else the same as in the previous example, with $\pi(q|\theta) = (1+\theta)q^{0.5}$.
So $F(\theta) = (\theta-1)/(2-1)$, $f(\theta) = 1/(2-1) =1$, $F(\theta)/f(\theta) = \theta-1$, for any θ in [1,2].
Then the principal's expected gains from trade are
$\int_1^2 [S(q(\theta)|\theta) - x(\theta)]\, f(\theta)\, d\theta = \int_1^2 \{S(q(\theta)|\theta) - q(\theta)[\theta+F(\theta)/f(\theta)]\}f(\theta)d\theta - U(x,q|2)$
$= \int_1^2 [(\theta+1)q(\theta)^{0.5} - q(\theta)(2\theta-1)]\, d\theta - [x(2)-2q(2)]$.
To maximizing the integrand at every θ, we want
$0 = 0.5(\theta+1)q^{-0.5} - (2\theta-1)$, which yields $q(\theta) = [0.5(\theta+1)/(2\theta-1)]^2$.
Because this $q(\theta)$ is monotone decreasing in θ, we know that it is actually feasible.
[If this $q(\theta)$ were increasing over any part of the interval, then it would not be feasible, and we would have an "irregular" case which is more complicated to solve (e.g: Myerson, 1981).]
To get $U(x,q|2) = 0$, we let $x(2) = 2q(2) = 2\times0.25 = 0.5$.
Then the information-rent equations give us
$U(x,q|\theta) = \int_\theta^2 q(t)\, dt = 0.25[0.25t + 0.75LN(2t-1) - 2.25/(4t-2)]\big|_\theta^2$,
$x(\theta) = \theta q(\theta) + \int_\theta^2 q(t)\, dt$, which yields $x(2) = 0.25$ and $x(1) = 1.456$

Auction: There are n possible suppliers of a service worth V to the buyer. Each supplier i has a cost type $\tilde{\theta}_i$ independently drawn from a Uniform distribution on an interval [A,B].

So $F(\theta_i) = (\theta_i - A)/(B-A)$, $f(\theta_i) = 1/(B-A)$, $F(\theta_i)/f(\theta_i) = \theta_i - A$, for any θ_i in [A,B].

An auction is planned, to select the supplier who actually sells to the buyer.

Let $q_i(\theta_1,\ldots,\theta_n)$ denote the conditional probability that i will sell to the buyer, and let $w_i(\theta_1,\ldots,\theta_n)$ denote the conditional expected payment to i, given these types $\tilde{\theta}_1=\theta_1,\ldots,\tilde{\theta}_n=\theta_n$.

Integrating over all types other than θ_i, let $Q_i(\theta_i)$ denote the marginal probability that i will sell, and let $W_i(\theta_i)$ denote the expected payment to i, given his own type $\tilde{\theta}_i = \theta_i$.

[If n=2 then $Q_1(\theta_1) = \int_A^B q_1(\theta_1,\theta_2)\, f(\theta_2)\, d\theta_2$, $W_1(\theta_1) = \int_A^B w_1(\theta_1,\theta_2)\, f(\theta_2)\, d\theta_2$.]

Expected payment to i is $EW_i(\tilde{\theta}_i) = U_i(B) + \int_A^B Q_i(t)[t_i + t_i - A]f(t_i)dt = U_i(B) + E[Q_i(\tilde{\theta}_i)(2\tilde{\theta}_i - A)]$

Suppose the buyer never pays more than B, so $U_i(w,q|B)=0$. Then the buyer's expected profit is

$\sum_i E\{Q_i(\tilde{\theta}_i)[V-(2\tilde{\theta}_i-A)]\} = \int_A^B \ldots \int_A^B \sum_i q_i(\theta_1,\ldots,\theta_n)[V-(2\theta_i-A)]\, f(\theta_1)\ldots f(\theta_n)\, d\theta_1\ldots d\theta_n$.

To maximize expected profit, buy from i (let q_i=1) only when $\theta_i = \min\{\theta_1,\ldots,\theta_n\} < (V+A)/2$.

Trading between a buyer and a seller, who knows more about the object being sold

Facts about Uniform distributions. Suppose that $\tilde{X}$ is a random variable drawn from a Uniform distribution on the interval from A to B, for some given numbers A and B such that $A < B$.
Then $E(\tilde{X}) = (A+B)/2$. Furthermore, for any number θ between A and B:
$\Pr(\tilde{X}<\theta) = \Pr(\tilde{X}\leq\theta) = (\theta-A)/(B-A)$,
$E(\tilde{X}|\tilde{X}\leq\theta) = E(\tilde{X}|\tilde{X}<\theta) = (A+\theta)/2$, $E(\tilde{X}|\tilde{X}\geq\theta) = E(\tilde{X}|\tilde{X}>\theta) = (\theta+B)/2$.

Example. To illustrate the problems of trading between individuals who have different information, consider the following simple situation, involving two individuals. Individual 1 is the seller of some unique object which he owns. Individual 2 is the only possible buyer of this object. Depending on the object's quality, it may be worth as little as \$40 to the seller and \$60 to the buyer (if its quality is low) or as much as \$100 to the seller and \$120 to the buyer (if its quality is high). The seller knows the quality of the object. The seller's type $\tilde{t}_1$ is his value of keeping the object. With any quality, the object would be worth \$20 more to the buyer than to the seller.
That is, given 1's type $\tilde{t}_1$, the value of the object to the buyer would be $S_2(t_1) = \tilde{t}_1+20$.
The buyer's belief about $\tilde{t}_1$ is described by a Uniform distribution on the interval \$40 to \$100.

Game where buyer bids Suppose first that the buyer can offer to buy for any positive price r, and then the seller will accept or reject the offer. If the offer is rejected then they each get profit 0.
If the offer is accepted then 1's profit is $r-\tilde{t}_1$ and 2's profit is $S_2(\tilde{t}_1)-r$.

In a subgame-perfect equilibrium, the seller will accept if $\tilde{t}_1 < r$, but the seller will reject if $\tilde{t}_1 > r$.
The buyer's expected profit from offering any price r is $Y(p) = \Pr(\tilde{t}_1<p)*\mathbf{(}E(V_2(\tilde{t}_1)|\ \tilde{t}_1<p) - p\mathbf{)}$.
For any number r between 40 and 100, this expected profit is
$U_2(r) = \Pr(\tilde{t}_1<p)*\mathbf{(}E(\tilde{t}_1+20|\ \tilde{t}_1<r) - r\mathbf{)} = \Pr(\tilde{t}_1<r)*\mathbf{(}E(\tilde{t}_1|\ \tilde{t}_1<r) + 20 - r\mathbf{)} =$
$\mathbf{(}(r-40)/(100-40)\mathbf{)}*\mathbf{(}(40+r)/2 + 20 - r\mathbf{)} = (r-40)*(80-r)/120 = (-3200 + 40r - r^2)/120$.
This quadratic formula is maximized by letting $r = 60$.
(The buyer cannot gain by bidding less than 40 or more than 100, because a bid below 40 would be surely rejected, and a bid above 100 would be worse than the surely-accepted bid of 100.)
So in the unique subgame-perfect equilibrium of this game, the buyer offers to buy for \$60, and the seller accepts if $\tilde{t}_1 < 60$. The probability of trade is Pr(trade) = (60−40)/(100−40) = 1/3.

Game where seller bids Suppose now that the seller can offer to sell for any positive price y, and then the buyer will accept or reject the offer. If the offer is rejected then they each get profit 0. If the offer is accepted, then 1's profit is $y-\tilde{t}_1$ and 2's profit is $S_2(\tilde{t}_1)-y$. In this game, the price is named by the person who has private information, and so signaling effects give us many equilibria. We may also reinterpret this as a market where 1 commits to a price at which he must sell or keep.

Let's look first for an equilibrium where there is some price r such that the buyer would surely accept an offer to sell for r but would surely reject an offer to sell for any price higher than r.
In this equilibrium, the seller will offer r if $\tilde{t}_1 < r$.
For player 2 to accept the offer r, 2's expected profit from accepting r must not be negative, so
$0 \le E(V_2(\tilde{t}_1)|\,\tilde{t}_1<r) - r = E(\tilde{t}_1+20|\tilde{t}_1<r) - r = (40+r)/2 + 20 - r$, which implies $r \le 80$.
For player 2 to reject any offer to sell at a price higher than r, such a trade must be unprofitable for player 2 when she makes the worst inference about player 1, which is that his type is 40, in which case the object would be worth $40+20 = \$60$ to player 2.
So we can construct such an equilibrium for any r such that $60 \le r \le 80$.
In such an equilibrium, types higher than r may be expected to make some offer higher than 120, which player 2 could never profitably accept.
An offer between r and 120 may be rejected by player 2 because this surprise offer may lead player 2 to believe that 1's type is 40, in which case the object is only worth 60 to player 2. Among these almost-pooling equilibria, player 1 most prefers the equilibrium with r = 80.
In this equilibrium, the probability of trade is $\Pr(\text{trade}) = \Pr(\tilde{t}_1<80) = (80-40)/(100-40) = 2/3$.
Reinterpreted in market: We'd get excess demand if r<80; the market-clearing equilibrium is r=80.

There are many other equilibria where 1's types make more offers.
Let's look for an equilibrium in which some types of player 1 would offer to sell for \$70, but all higher types would offer to sell for \$100, and player 2 would be sure to accept \$70 but her probability of accepting \$100 would be between 0 and 1. To find this equilibrium, we have two unknowns to find: let q denote the probability that player 2 would accept an offer of \$100, and let θ denote the highest type of player 1 that would offer \$70.
For player 2 to be willing to randomize between accepting and rejecting \$100,
her expected profit from accepting it must be 0, and so
$0 = E(S_2(\tilde{t}_1)|\tilde{t}_1>\theta) - 100 = E(\tilde{t}_1+20|\tilde{t}_1>\theta) - 100 = (\theta+100)/2 + 20 - 100$, and so $\theta = 60$.
For player 1 to offer \$70 below when his type is below θ but \$100 when his type is above θ, we need that $70-t_1 \ge q(100-t_1)$ when $t_1<\theta$, and $70-t_1 \le q(100-t_1)$ when $t_1>\theta$.
These inequalities imply $70-\theta = q(100-\theta)$, and so $q = (70-60)/(100-60) = 1/4$.
In this equilibrium, $\Pr(\text{trade}) = \Pr(\tilde{t}_1<\theta) + \Pr(\tilde{t}_1>\theta)q = (20/60) + (40/60)(1/4) = 1/2$.
This is also a market equilibrium, with no excess demand at either price: $E(S_2(\tilde{t}_1)|\tilde{t}_1\le\theta) = 70$.

There is a separating equilibrium in which each possible type t_1 of player 1 would offer to sell for $y = t_1+20$, and the probability of player 2 accepting would depend on the offer y according to the formula $\hat{Q}(y) = e^{-(y-60)/20}$, for any $y \ge 60$. [*Derivation:* For $y = t+20$ to maximize $\hat{Q}(y)(y-t)$, we need $0 = \hat{Q}'(y)(y-t) + \hat{Q}(y)$ when $y = t+20$, and so $-1/20 = \hat{Q}'(y)/\hat{Q}(y) = d/dy\, LN(\hat{Q}(y))$.]
So the probability of trade depends on 1's type t by the function $q(t) = \hat{Q}(t+20) = e^{-(t-40)/20}$, and the expected payment to type t is $x(t) = (t+20)q(t)$. This plan is <u>safe</u> in the sense that the buyer does not lose from trading with any type of seller. It is the best safe plan for all types of the seller.

Let's consider **a bilateral trading problem where the seller has a single indivisible asset to sell, and the seller has two possible cost types**, but the buyer has no private information.
To be specific, consider an example where the seller's cost type is either θ_L=20 or θ_H=40, and the buyer's value for the asset is S_L = 30 if the seller's type is θ_L, but is S_H = 50 if the seller is θ_H.
(So the asset is always worth 10 more to the buyer than to the seller.)
For now, let's keep the probability of the high type p_H as a parameter, with $p_L=1-p_H$.

In a trading plan (x,q), for each seller-type t, q_t is t's probability of selling, x_t is t's expected revenue.
In any such plan, a high-type seller's expected gain is $U_H = x_H - \theta_H q_H$, a low-type seller's expected gain is $U_L = x_L - \theta_L q_L$, and the buyer's expected gain is $V = p_H(S_H q_H - x_H) + p_L(S_L q_L - x_L)$.

In such bilateral trading problems with a single asset and one-sided private information, the uninformed buyer's optimal trading plan is always to offer a fixed take-it-or-leave it bid that is equal to (or slightly more than) one of the possible cost-types of the seller.
That is, the buyer's optimal incentive-compatible plan could be either to bid θ_L=20, so that $q_L=1, x_L=20$, $q_H=0$, $x_H=0$ (separating); or to bid θ_H=40, so that $q_L=1=q_H$, $x_L=40=x_H$ (pooling).
The pooling offer is better for the buyer than the separating offer when
$p_H\, 50 + p_L\, 30 - 40 \geq p_L(30-20)$, that is, when $p_H \geq 2/3$ (as $p_L=1-p_H$).

[To verify this optimality, notice that the H-participation and H|L-incentive constraints are binding in both of these trading plans. So we can do Lagrangean analysis with λ_H=1, $\alpha_{H|L} = p_L$. Then:
$\mathcal{L}(x,q;\lambda,\alpha) = [p_H(S_H q_H - x_H) + p_L(S_L q_L - x_L)] + \lambda_H(x_H - \theta_H q_H) + \alpha_{H|L}[x_L - \theta_L q_L - x_H + \theta_L q_H]$
$\quad = p_H q_H\{S_H - [\theta_H + (\theta_H - \theta_L)p_L/p_H]\} + p_L q_L(S_L - \theta_L)$
This Lagrangean needs to be maximized over q_L and q_H subject to $0 \leq q_L \leq 1$ and $0 \leq q_H \leq 1$.
With $S_L > \theta_L$, q_L=1 achieves this maximum. The maximizing value of q_H is 1 or 0 depending on whether S_H=50 is greater or less than $[\theta_H + (\theta_H - \theta_L)p_L/p_H] = 40 + (40-20)p_L/p_H$.
But $S_H \geq [\theta_H + (\theta_H - \theta_L)p_L/p_H]$ is algebraically equivalent to $p_H S_H + p_L S_L - \theta_H \geq p_L(S_L - \theta_L)$, which holds when the buyer prefers the offer-θ_H pooling plan (buying from both types) over the offer-θ_L separating plan (buying only from L).]

But now, what trading plan might be used when the informed seller has the all the market power?
In such problems (where sellers have private information, and where buyers have no private information but will participate in the market only if expected gains from trade are nonnegative), we may say that a trading plan is safe (safely profitable for the uninformed buyers) iff the buyers get nonnegative expected profits from each type of seller.
So a safe trading plan (x,q) here satisfies the safety constraints: $S_H q_H - x_H \geq 0$, $S_L q_L - x_L \geq 0$.
There may be many safe incentive-compatible trading plans that satisfy these safety constraints and the informational incentive constraints.

Fact. All types of the informed seller can agree on which safe incentive-compatible plan is best.
[To see why, suppose to the contrary that the best safe incentive-compatible plan for type H was (x,q) but the best safe incentive-compatible plan for type L was $(\hat{x},\hat{q})$. Then let the plan $(\bar{x},\bar{q})$ coincide with (x,q) for type H but coincide with $(\hat{x},\hat{q})$ for type L. Then

$\bar{x}_H - \theta_H \bar{q}_H = x_H - \theta_H q_H$ (by definition of the $(\bar{x},\bar{q})$ plan when the type is H)

$\geq \hat{x}_H - \theta_H \hat{q}_H$ (because H prefers (x,q) over $(\hat{x},\hat{q})$)

$\geq \hat{x}_L - \theta_H \hat{q}_L$ (because $(\hat{x},\hat{q})$ is incentive compatible)

$= \bar{x}_L - \theta_H \bar{q}_L$ (by definition of the $(\bar{x},\bar{q})$ plan when the type is L)

Also, $\bar{x}_L - \theta_L q_L = \hat{x}_L - \theta_L \hat{q}_L \geq x_L - \theta_L q_L \geq x_H - \theta_L q_H = \bar{x}_H - \theta_L \bar{q}_L$, so $(\bar{x},\bar{q})$ is incentive compatible.
$(\bar{x},\bar{q})$ is safe because $S_H \bar{q}_H - \bar{x}_H = S_H q_H - x_H \geq 0$ and $S_L \bar{q}_L - \bar{x}_L = S_L \hat{q}_L - \hat{x}_L \geq 0$.
But each type of seller gets the same expected payoff from $(\bar{x},\bar{q})$ as the other plans that was assumed best for it, and so $(\bar{x},\bar{q})$ is best for both types of seller among all safe incentive-compatible plans.]

So we can talk about the <u>best safe</u> plan for the seller, that is, the plan that is best for all types of seller among all plans that are incentive compatible and safely profitable for the buyer with all types.
In our example, it is easy to see that (q_L=1, x_L=30, q_H=1/3, x_H=50/3) is the best safe plan, given any probability p_H between 0 and 1. Having q_L=1 and x_L=30 gives type L the highest expected profits subject to the constraint that the buyer cannot pay more than $S_L = 30$.
Then the best safe plan for H must have x_H,q_H to maximize $x_H - 40q_H$ subject to $50q_H - x_H \geq 0$ and $x_H - 20q_H \leq 30-20$. The solution has $x_H = 50q_H$ and $(50-20)q_H = 30-20$ so $q_H = 1/3$.
Thus, the best safe plan is: $x_L = 30q_L$, q_L=1, $x_H = 50q_H$, $q_H = (30-20)/(50-20) = 1/3$,
so U_L=30−20, U_H=(50−40)/3.

A trading plan is <u>interim dominated</u> if there is some other feasible plan that would yield higher expected gains to each possible type of each individual (given only what he knows in his type).
Here, the seller has two possible types and the buyer has only one type, and so a plan (x,q) would be interim dominated by some other plan $(\hat{x},\hat{q})$ if $U_H < \hat{U}_H$, $U_L < \hat{U}_L$ and $V < \hat{V}$
[where $\hat{U}_H = \hat{x}_H - \theta_H \hat{q}_H$, $\hat{U}_L = \hat{x}_L - \theta_H \hat{q}_L$, and $\hat{V} = p_H(S_H \hat{q}_H - \hat{x}_H) + p_L(S_L \hat{q}_L - \hat{x}_L)$].
A trading plan (x,q) is <u>incentive efficient</u> if it is incentive compatible and it is not interim dominated by any other incentive-compatible plan.
In this example, the best safe plan is incentive-efficient when $p_H \leq 2/3$.
But when $p_H > 2/3$, this plan is interim dominated by a pooling plan
$\hat{q}_L = \hat{q}_H = 1$, $\hat{x}_L = \hat{x}_H = 50p_H + 30(1-p_H) = 30+20p_H > 43.333...$ when $p_H > 2/3$.

Now consider a market where there are many competitive buyers, any one of whom could use all the sellers' supply. So buyers' competition should drive their expected profits to 0 in equilibrium.
If $p_H \leq 2/3$, then the best safe plan (q_L=1, x_L=30, q_H=1/3, x_H=50/3) is incentive-efficient and may be considered a competitive equilibrium for this market. But what if p_H>2/3?
For example, suppose $p_H = 0.8$, so $50p_H+30(1-p_H) = 46$.
If all buyers were expected to offer this best-safe plan, then a deviating buyer who offered
($\hat{q}$=1, 46>$\hat{x}$≥43.34) could attract all sellers and make a positive profit on average.
But if all buyers competitively offered the break-even pooling plan (q=1,x=46), then a deviating buyer could gain by attracting H-sellers with $(\hat{q},\hat{x})$=(0.9,48$\hat{q}$), so $\hat{x}-20\hat{q}<26$, $\hat{x}-40\hat{q}>6$, $\hat{x}<50\hat{q}$.
The right definition of equilibrium in such markets has remained unclear.

Signalling. In the above example, suppose now that there is a signal that sellers could make which would cost $c_L>0$ for L-types but would cost $c_H>0$ for H-types.
Suppose there are excess buyers in the market, and the terms of trade that these competitive buyers offer the sellers will depend only on what the buyers observe about the seller's signal.
So for each observable signal-value, the competitive buyers should just break even, paying exactly the value of the asset conditional on the signal.

For what values of c_H and c_L could there be an equilibrium where both types choose to signal?
If both types are expected to signal, then they can sell the asset for $p_H S_H + p_L S_L$.
The worst that buyers could infer about a seller is that his type is L, in which case they would pay up to $S_L=30$ for his object. So a seller always has two alternatives to signaling: he can sell it for $S_L=30$, or he can keep the asset himself and get zero gains from trade.
Thus, for both typs of seller to actually want to use this signal in an equilibrium, we must have
$(p_H S_H + p_L S_L) - \theta_H - c_H \geq \max\{S_L - \theta_H, 0\}$ and $(p_H S_H + p_L S_L) - \theta_L - c_L \geq \max\{S_L - \theta_L, 0\}$.
In this example, this becomes $p_H 50 + (1-p_H)30 - 40 - c_H \geq 0$, $p_H 50 + (1-p_H)30 - 20 - c_L \geq 30-20$,
and so we must have $20p_H - 10 \geq c_H$ and $20p_H \geq c_L$.
The former condition cannot be satisfied by any positive c_H unless $p_H > 1/2$.
But when $p_H = 0.8$, for example, this pooling scenario is an equilibrium if $c_H \leq 6$ and $c_L \leq 16$.

For what values of c_H and c_L could there be an equilibrium where the seller would choose to signal when his type is H but not when his type is L?
When the low type is separated, its competitive bids from buyers will be $S_L = 30$, and so type-L sellers will get expected gain $S_L - \theta_L = 30-20 = 10$ from their nonsignaling separation.
But signaling in this separating scenario becomes evidence of type H, and so those sellers who signal can get competitive bids of $S_H = 50$ from the buyers.
So for the low types to not signal, we must have $S_H - \theta_L - c_L \leq S_L - \theta_L$.
For the high types to signal, we must have $S_H - \theta_H - c_H \geq \max\{0, S_L - \theta_H\}$.
So this separating scenario is an equilibrium when $50-20-c_L \leq 30-20$ and $50-40-c_H \geq 0$,
that is, when $c_L \geq 20$ and $c_H \leq 10$.