

An impossibility theorem of social choice

<http://home.uchicago.edu/~rmyerson/research/schch1.pdf>

Can a political institution abolish multiple equilibria?

A variant of Arrow's impossibility theorem says No.

Let N denote a given set of individual voters.

Let Y denote a given set of social-choice options, of which the voters must select one.

We assume that N and Y are both nonempty finite sets.

Let $L(Y)$ denote the set of strict transitive orderings of the alternatives in Y .

Let $L(Y)^N$ denote the set of profiles of preference orderings, one for each voter.

We may denote such a preference profile by a profile of utility functions $u = (u_i)_{i \in N}$, where each u_i is in $L(Y)$. So if the voters' preference profile is u , then the inequality $u_i(x) > u_i(y)$ means that voter i prefers alternative x over alternative y .

($u_i(x) = \#\{y \in T \mid x \text{ is preferred to } y \text{ under } i's \text{ preference in } u\}$.)

A social choice function is any function $F: L(Y)^N \rightarrow Y$, where $F(u)$ denotes the alternative in Y to be chosen if the voters' preferences were as in u .

Let $F(L(Y)^N) = \{F(u) \mid u \in L(Y)^N\}$.

Given any game form $H: \times_{i \in N} S_i \rightarrow Y$ (where each S_i is a nonempty strategy set for i), let $E(H, u)$ be the pure Nash equilibrium outcomes of H with preferences u . That is,

$$E(H, u) = \{H(s) \mid s \in \times_{i \in N} S_i, \text{ and, } \forall i \in N, \forall r_i \in S_i, u_i(H(s)) \geq u_i(H(s_{-i}, r_i))\}.$$

Theorem (Muller-Satterthwaite) Suppose that a social choice function $F: L(Y)^N \rightarrow Y$ and a game form $H: \times_{i \in N} S_i \rightarrow Y$ satisfy

$$\#F(L(Y)^N) > 2 \text{ and } E(H, u) = \{F(u)\} \quad \forall u \in L(Y)^N.$$

Then there is some h in N such that $u_h(F(u)) = \max_{x \in F(L(Y)^N)} u_h(x), \forall u \in L(Y)^N$.

That is, if an institution H admits more than two possible outcomes and always yields a unique pure-strategy Nash equilibrium, then H must be a dictatorship.

Different democratic institutions may have very different sets of equilibria, but we cannot expect any to abolish multiplicity or randomization of Nash equilibria,

and so democratic outcomes may depend on more than just the voters' preferences.

Lemma (monotonicity) For any pair of preference profiles u and v in $L(Y)^N$,

if $x \in E(H, u)$ and $\{(i, y) \in N \times Y \mid v_i(y) > v_i(x)\} \subseteq \{(i, y) \in N \times Y \mid u_i(y) > u_i(x)\}$, then $x \in E(H, v)$.

Example: the Condorcet cycle. Social options are $Y = \{a, b, c\}$, voters are $N = \{1, 2, 3\}$.

$u_1(a) = 2 > u_1(b) = 1 > u_1(c) = 0$; $u_2(b) = 2 > u_2(c) = 1 > u_2(a) = 0$; $u_3(c) = 2 > u_3(a) = 1 > u_3(b) = 0$.

If H is symmetric with respect to social options (neutrality) and voters (anonymity) then its pure-strategy equilibrium outcomes are either $E(H, u) = Y$ (multiple equilibria) or $E(H, u) = \emptyset$ (only randomized equilibria).

The Probabilistic Voting model: a simple formulation

[For sophisticated probabilistic voting models that also include campaign contributions, see Persson and Tabellini Political Economics (2000) chapters 3 and 5. See also G. Grossman and E. Helpman, "Electoral competition and special interest politics," Review of Economic Studies 63:265-286 (1996), for a model that includes a version of probabilistic voting and campaign contributions.]

Let Y denote the set of social-choice alternatives or policy options for the government.

There are two parties, and each party k in $\{1,2\}$ can simultaneously choose a policy x_k in Y .

Let us also allow that a party could promise to choose its policy according to any probability distribution σ_k in $\Delta(Y)$.

Each voter has a policy-type i that is independently drawn from a set of types I , getting type i with probability r_i . Each policy y in Y gives some utility $u_i(y)$ to every type- i voter.

In addition, each voter has a net personal bias toward party 1 that is drawn independently from a uniform distribution on the interval $[-\delta, \delta]$. A voter of type i with policy-type i and bias β gets payoff $\beta + u_i(x_1)$ if party 1 wins, but gets payoff $u_i(x_2)$ if party 2 wins.

After the parties have chosen their policy positions x_1 and x_2 , each voter votes for the party that offers him the higher payoff, given his policy-type and his bias.

Each party wants to maximize its probability of winning the majority-rule election.

Fact. If both parties choosing the same policy $x_1 = x_2 = x \in Y$ is an equilibrium, then x maximizes the expected sum of the voters' utility $x \in \operatorname{argmax}_{y \in Y} \sum_{i \in I} r_i u_i(y) = \operatorname{Eu}_i(y)$.

Proof. When they both choose x for sure, a voter of any policy-type is equally likely to vote for either party, and so each party has an equal probability of winning a majority of the vote.

Now, keeping party 2 at x for sure, suppose that party 1 deviated and promised to choose x with probability $1 - \varepsilon$ and some other y with probability ε , given $\varepsilon > 0$ and $y \in Y$.

The possibility of changing policy from electing 1 instead of 2 would change type- i 's expected utility by the amount $\varepsilon(u_i(y) - u_i(x))$, and so a type- i voter will vote for 1 if his bias β satisfies $\beta + \varepsilon(u_i(y) - u_i(x)) > 0$, that is $\beta > -\varepsilon(u_i(y) - u_i(x))$, which has probability $1/2 + \varepsilon(u_i(y) - u_i(x))/(2\delta)$, if ε is small enough so that this formula is between 0 and 1.

So when ε is small, the probability of any randomly-sampled voter voting for party 1 is $1/2 + \varepsilon \sum_{i \in I} r_i (u_i(y) - u_i(x))$.

Thus, if $\sum_{i \in I} r_i u_i(y) > \sum_{i \in I} r_i u_i(x)$ then the ε -probabilistic deviation from x to y would make any randomly sampled voter more likely to vote for party 1 than for party 2, and so (be different voters' votes are independent) the deviating party 1 would get a greater than 1/2 chance of winning the election. But in equilibrium, such deviations from x cannot increase a party's chances of winning, and so we must have $\sum_{i \in I} r_i u_i(x) \geq \sum_{i \in I} r_i u_i(y)$ for all y in Y .

This result tells us that a convergent pure equilibrium must choose a policy that is a utilitarian optimum, maximizing the expected total utility of all voters.

But this result is somewhat misleading, because such convergent pure equilibria do not generally exist. They exist only when δ is very large, that is, when the effect of policy is small relative to the effect of individuals' biases toward one party or the other.

For example, suppose that $Y=\{a,b,c\}$, $I = \{1,2,3\}$, and $u_i(y)$ is as follows

Type i	$u_i(a)$	$u_i(b)$	$u_i(c)$	r_i
1	2	1	0	0.4
2	0	2	1	0.3
3	1	0	2	0.3
Eu_i :	1.1	1.0	0.9	

So the utilitarian-optimum result is that, if there is a convergent equilibrium where both parties choose the same policy x , it must be the policy $x=a$, which maximizes voters' expected utility.

But when δ is small, if there are many voters, then policy a almost-surely beats policy b , policy b almost-surely beats policy c , and policy c almost surely beats policy $[a]$, and either party could find a promise that would win with probability greater than 1/2 if it knew what the other party's (possibly probabilistic) promise would be. (Any surely promised randomization in $\Delta(Y)$ could be beaten by another promised randomization that shifts probability from b to a or from c to b or from a to c .)

In the limiting case of $\delta=0$, this case reduces to the Condorcet cycle [ABC cycle] which a unique equilibrium where both parties choose policies randomly in the bipartisan set $\{a,b,c\}$ as defined by Laffond, Laslier, and Le Breton (1993); see also my Fundamentals of Social Choice Theory survey paper at <http://home.uchicago.edu/~rmyerson/research/schch1.pdf>

So for small δ , there is no convergent equilibrium where both parties make the same predictable promise. To have a pure convergent equilibrium at policy a here, we must have $\delta > 1.5$.

To see that convergent equilibrium at a requires $\delta > 1.5$, consider party 1 deviating to put probability ϵ on policy c , while party 2 remains at policy a for sure.

60% of the voters prefer c over a , but the 40% type-1s who prefer a care twice as much, and so the fraction of voters whom party 1 gains $0.6(1\epsilon)/\delta$ is less than the fraction $0.4(2\epsilon)/\delta$ that party 1 loses by the deviating. But this calculation goes wrong when ϵ becomes large enough that $(2\epsilon)/\delta > 1/2$, because then party 1 will have lost all of the type-1 voters, and then further increases in ϵ can win more voters without losing any more voters. So the equilibrium might be overturned by $\epsilon=1$ (all probability on c). That is, consider party 1 deviating to c for sure.

The least net pro-1 bias for a type- i voter to support party 1 is then $u_i(a)-u_i(c)$, and so a type- i voter's probability of voting for party 1 is $\max\{0, \min\{1, (\delta - (u_i(a)-u_i(c)))/(2\delta)\}\}$.

So in the whole population, the probability of a voter voting for the deviating party 1 is

$$0.4\max\{0,\min\{1,(\delta-2)/(2\delta)\}\} + 0.3\max\{0,\min\{1,(\delta+1)/(2\delta)\}\} + 0.3\max\{0,\min\{1,(\delta+1)/(2\delta)\}\}$$

$$= 0.4\max\{0,(\delta-2)/(2\delta)\} + 0.3\min\{1,(\delta+1)/(2\delta)\} + 0.3\min\{1,(\delta+1)/(2\delta)\}.$$

When $\delta < 1$, this probability of voting for 1 becomes $0+(0.3+0.3)(1) = 0.6 > 1/2$, and so the equilibrium fails.

When $\delta > 2$, this probability of voting for 1 is $1/2 - 0.4(1/\delta) + (0.3+0.3)(0.5/\delta) = 1/2 - 0.1/\delta < 1/2$, and so the equilibrium does not fail.

When $1 \leq \delta \leq 2$, this probability of voting for 1 is $(0.3+0.3)(1/2 + 1/(2\delta))$ which is $> 1/2$ when $\delta < 1.5$. So the equilibrium fails when $\delta < 1.5$.

Citizen-Candidate Model T. Besley and S. Coate, "An economic model of representative democracy," Quarterly J of Economics 112:85-114 (1997).

$N = \{\text{citizens}\}$, $Y = \{\text{policy space}\}$. For each $i \in N$, $u_i: Y \rightarrow \mathbb{R}$ is i's utility for policies.

δ = cost of becoming a candidate. Let θ_i be i's ideal point $\theta_i = \operatorname{argmax}_x u_i(x)$.

First, each citizen decides independently whether to become a candidate.

Then all citizens learn $K = \{\text{candidates}\} \subseteq N$, and each votes for one candidate.

The candidate with the most votes is the winner (ties resolved by randomization) and the government policy is the ideal point of the winner.

So if j is winner then each citizen i gets payoff $u_i(\theta_j)$ if $i \notin K$, or $u_i(\theta_j) - \delta$ if $i \in K$.

If $K = \emptyset$ then the outcome is some given x_0 in Y , and i gets $u_i(x_0)$.

The game is analyzed by looking at subgame-perfect equilibria in pure (nonrandom) strategies, after eliminating dominated strategies (voting for the least-preferred candidate) in each subgame after K is determined. The existence of such equilibria can be proven.

We consider cost δ to be small, taking limit as $\delta \rightarrow 0$.

An equilibrium in which exactly one candidate enters can only be near or at (as $\delta \rightarrow 0$) a Condorcet-winning policy position. Equilibria where exactly two candidates i and j enter (Duverger's equilibria) can exist for any $\{i,j\}$ such that the number of citizens who prefer i 's ideal policy over j 's is equal to the number who prefer j 's ideal over i 's.

Equilibria with three or more tied winners are hard to sustain (for the same reason as in Feddersen AJPS 1992): If a pure-strategy equilibrium generates a tie among k candidates, then no voter can strictly prefer any two of the tied candidates over the k -way randomization, because he could break the tie in favor of whichever candidate he was not expected to vote for (in the eqm).

But we can construct equilibria where three or more candidates enter even though most are expected to lose, because the presence of these spoilers can change the focal equilibrium in the subgame after candidates' entry. Remember, for any pair of candidates, there exists an equilibrium in the plurality-voting election where this pair is considered to be the only serious race, and so everybody votes for the one in this pair whom he prefers.

Consider a simple Hotelling example where $Y = [0,100]$, citizens have ideal points that are distributed uniformly over the interval 0 to 100, and each citizen's policy-payoff is minus the distance of policy from his ideal point.

Pick any x such that $2 < x < 98$. We can construct an equilibrium in which seven candidates enter with ideal points $\{0, 1, 2, x, 98, 99, 100\}$.

On the equilibrium path, the only serious race is 0 versus x , and x wins.

But if any candidate other than x dropped out, then the post-entry subgame equilibrium would switch to one where the only serious race is between two extreme candidates, the least moderate remaining on the side of the unexpected dropout, and the most moderate on the other side.

(E.g.: if 0 dropped out, then the serious race would be between 1 and 98, and 98 would win).

An unexpected extra entrant could be ignored (or could lead to an eqm where the 2 or 98 wins, whichever is worse for the unexpected entrant).

A model of leaders and supporters in contests for power

(R, λ, s, c, δ)

The Autocrat's Credibility Problem and Foundations of the constitutional State

<http://home.uchicago.edu/~rmyerson/research/foundatn.pdf>

An island principality yields income R that can be consumed or allocated by the ruler.

The ruler is the leader who won the most recent battle on the island.

Battles occur whenever a new challenger arrives, at a Poisson rate λ.

(In any time interval ε, P(challenger arrives) = $1 - e^{-\lambda \epsilon} \approx \lambda \epsilon$ if $\epsilon \approx 0$.)

A leader needs support from captains to have any chance of winning a battle.

$\Pr(\text{leader with } n \text{ captains wins against a rival with } m \text{ captains}) = p(n|m) = n^s / (n^s + m^s)$.

Let c denote a captain's cost of supporting a leader in battle.

The prince and the captains are assumed to be risk neutral and have discount rate δ.

Consider a leader who has n supporters, but expects all rivals to have m supporters.

(For simplicity, we will always assume stationary expectations about rivals.)

If the leader has promised to give each supporter an income y (as long as the leader rules)

then, when there is no challenger, a supporter's expected discounted payoff is

$$U(n,y|m) = (y - \lambda c) / [\delta + \lambda - \lambda p(n|m)].$$

For these captains to rationally give support in battle, we need $p(n|m)U(n,y|m) - c \geq 0$.

The lowest income y satisfying this participation constraint is $Y(n|m) = (\delta + \lambda)c / p(n|m)$

The leader's expected discounted payoff is:

$V(n,y|m) = (R - ny) / [\delta + \lambda - \lambda p(n|m)]$ when he rules with no immediate challenge,

$W(n,y|m) = p(n|m) V(n,y|m) = p(n|m)(R - ny) / [\delta + \lambda - \lambda p(n|m)]$ on the eve of battle.

An absolute monarch is one who is released from all constraints of law.

An absolute leader who cheated a supporter would not be punished by anyone else,

although of course the cheated individual might be less likely to support him in the future.

(An absolutist would have no incentive to pay supporters if even those cheated don't react.)

So a leader is absolute when his relationships with all supporters are purely bilateral,
as if supporters have no communication with each other.

Against m, a force of n captains is feasible for an absolute leader iff

there exists some wage rate y such that $y \geq Y(n|m)$ and $V(n,y|m) \geq V(k,y|m) \quad \forall k \in [0,n]$.

First is participation constraint for captains, second is absolutist's moral-hazard constraint.

Let $v(n|m) = V(n,Y(n|m)|m) = [R - nc(\delta + \lambda) / p(n|m)] / [\delta + \lambda - \lambda p(n|m)]$,

and let $w(n|m) = W(n,Y(n|m)|m) = [p(n|m)R - nc(\delta + \lambda)] / [\delta + \lambda - \lambda p(n|m)]$.

Proposition 1. If $n > 0$ and y satisfy the feasibility condition for an absolute leader against m, then
there exist $k > n$ such that $v(k|m) > V(n,y|m)$ and $w(k|m) > W(n,y|m)$.

Proof. [Easy if $y > Y(n|m)$.] $Y'(n|m) < 0$. $\text{AbsFeas} \Rightarrow V'(n,y|m) \geq 0$. [$'$ = deriv wrt 1st.]

So with $y = Y(n|m)$, $v'(n|m) = V'(n,y|m) - Y'(n|m)n / [\delta + \lambda - \lambda p(n|m)] > 0$.

So an absolute leader could always benefit by commitment to maintain a larger force.

Now suppose captains communicate at court, and a complaint by any captain could switch them to a distrustful equilibrium, where nobody trusts the ruler to reward supporters.

Complaining-only-if-cheated is incentive compatible, as captains expect $U>0$ on eqm path.

With challenges at rate λ and no support, the ruler's expected payoff would be $R/(\delta+\lambda)$.

So we say n is feasible for a leader with a weak court against m iff $v(n|m) \geq R/(\delta+\lambda)$.

$V(0,y|m) = R/(\delta+\lambda)$, so feasible for absolutist \Rightarrow feasible for leader with a weak court.

This court is called "weak" because it cannot change the arrival rate of new challengers.

But when a ruler is known to have no support, immediate challenges may be more likely.

Then loss of confidence at court could lead to a rapid downfall of the leader.

So we say n is feasible for a leader with a strong court against m iff $v(n|m) \geq 0$.

Proposition 2. Suppose that n is feasible for a leader with a weak court against m .

Then $nY(n|m)/R \leq p(n|m)\lambda/(\delta+\lambda)$ and $n \leq R\lambda p(n|m)^2/[c(\delta+\lambda)^2]$.

If $n > 0$ and $s > 0.5$ then $m \leq M_0 = [R\lambda(2s-1)^{2-1/s}]/[4s^2c(\delta+\lambda)^2]$.

We may say that a force size m is globally feasible for leaders of some kind (absolute, or with weak courts, or with strong courts) iff m is feasible against m for such leaders.

Proposition 3. Suppose that $s \geq 2/3$.

If n is feasible against m for a weak-court leader and $0 < n \leq m$, then $w'(n|m) > 0$.

So if m is globally feasible for weak-court leaders then $\text{argmax}_{k \geq 0} w(k|m) > m$.

We may say that m is a negotiation-proof equilibrium iff $w(m|m) = \max_{n \geq 0} w(n|m)$, so that any new leader before first battle would want to negotiate the same force size. By Prop 3, such a negotiation-proof eqm cannot be globally feasible with weak courts.

Proposition 4. When $s \leq 2$, the negotiation-proof equilibrium is $m_1 = Rs/[c(4\delta+2\lambda+s\lambda)]$.

In this eqm, supporters get the fraction $m_1 Y(m_1|m_1)/R = 2s(\delta+\lambda)/(4\delta+2\lambda+s\lambda)$ [$\rightarrow 1$ as $s \rightarrow 2$].

When $s \geq 0.763$, this equilibrium m_1 is greater than the bound M_0 from Proposition 2, and so an absolutist or a leader with a weak court could not get any support against this eqm.

The courtiers are in a coordination game with multiple equilibria.

With $s > 1$, nobody should support a leader who is not supported by anybody else (p too small).

Each wants to support the leader as long as he trusts the leader and enough others are expected to also support the leader. Once a critical mass of supporters has been gathered, before the first battle for power, the leader's speech could make focal the equilibrium with trust among the w -maximizing m supporters.

Short proof of the weak M+1 law for SNTV with a large Poisson electorate

Suppose that there are K candidates, numbered $1, 2, \dots, K$, in an election with single nontransferable vote where the top M candidates win. Here $M < K$.

In case of a tie for M 'th and $M+1$ 'th place, a random ordering of the candidates is generated, and the set of M winners is completed by selecting from the borderline-winning candidate in this order.

Each voter has a type which is drawn independently from some finite set according to some fixed probability distribution.

Each type of voter has a strict utility ranking of the candidates, with $u_i(t)$ denoting the utility of candidate i winning for a voter of type t .

A voter's payoff from the election is the sum of the winners for him.

In the voting game n , the number of voters is a Poisson random variable with mean n . If W denotes the actual number of voters: $P(W=k) = e^{-n} n^k / k!$

A large equilibrium is a convergent sequence of equilibria of these games as $n \rightarrow \infty$.

By convergent, we mean that the expected fraction of the electorate who vote for each candidate i is converging to some limit τ_i . [$\tau_i = \sum_t r(t)\sigma(i | t)$.]

Choosing a subsequence if necessary, we may assume that other probabilities are also convergent to well-defined limits as $n \rightarrow \infty$.

Now consider a large equilibrium. Without loss of generality, we may assume that the candidates are numbered so that $\tau_1 \geq \tau_2 \geq \dots \geq \tau_K$.

Let X_i denote the number of votes for candidate i .

Then X_i is a Poisson random variable with mean $n\tau_i$, independent of other X_j .

The $\{i,j\}$ race is close when adding one vote for i or j could make one of them replace the other in the set of winners.

If there is no close race, then adding one more vote in cannot matter to anybody.

In equilibrium, each voter must vote cast the ballot that would maximize his conditional expected utility gain, relative to not voting, given that there is at least one close race.

A race between two candidates is serious iff its conditional probability of being close, given that there is some close race, is strictly positive in the limit as $n \rightarrow \infty$.

A candidate is serious iff he is involved in at least one close race.

A voter's conditional expected gain, given that there is a close race, from voting for his favorite serious candidate would be strictly positive in the limit.

So each voter's ballot in equilibrium must give him a strictly positive conditional expected gain, given that there is a close race.

Thus, in the large equilibrium, nobody votes for candidates who are not serious; that is, if h is not a serious candidate then $\tau_h = 0$.

From any standard paper on Poisson voting games, we get:

Fact. For any two candidates i and j such that $\tau_i > \tau_j$, the magnitude of the event " $1+X_j \geq X_i$ " is $\lim_{n \rightarrow \infty} \text{LN}(P(1+X_j \geq X_i | \tau, n))/n = 2(\tau_i \tau_j)^{0.5} - \tau_i - \tau_j = -(\tau_i^{0.5} - \tau_j^{0.5})^2$.

Here is a sketch of the argument:

If Y is a Poisson random variable with large mean m , and $k=\alpha m$ is an integer, then $P(Y=k) = e^{-m} m^k / k! \approx e^{-m} m^k / [(k/e)^k (2\pi k)^{0.5}]$

$$= e^{-m} m^{\alpha m} e^{\alpha m} / [(\alpha m)^{\alpha m} (2\pi \alpha m)^{0.5}] = e^{m(\alpha-1-\alpha \text{LN}(\alpha))} / (2\pi \alpha m)^{0.5}.$$

(Stirling's approximation for $k!$ is used here.)

Thus, $\text{LN}(P(Y=\alpha m))/m \approx \alpha-1-\alpha \text{LN}(\alpha)$.

$$\begin{aligned} \text{LN}(P(X_1=\beta n=X_2 | n))/n &= \sum_{i \in \{1,2\}} \tau_i \text{LN}(P(X_i=(\beta/\tau_i)\tau_i n)) / (\tau_i n) \\ &\approx \tau_1 [\beta/\tau_1 - 1 - (\beta/\tau_1) \text{LN}(\beta/\tau_1)] + \tau_2 [\beta/\tau_2 - 1 - (\beta/\tau_2) \text{LN}(\beta/\tau_2)] \\ &= 2\beta - \tau_1 - \tau_2 - 2\beta \text{LN}(\beta) + \beta \text{LN}(\tau_1) + \beta \text{LN}(\tau_2). \end{aligned}$$

This is maximized when $\text{LN}(\beta) = \text{LN}(\tau_1) + \text{LN}(\tau_2)$ and so $\beta = (\tau_1 \tau_2)^{0.5}$.

Thus, the magnitude of the event that X_1 and X_2 are equal (or close) is

$$\begin{aligned} \text{LN}(P(X_1=X_2 | n))/n &\approx \max_{\beta \geq 0} \text{LN}(P(X_1=\beta n=X_2 | n))/n \\ &\approx 2(\tau_1 \tau_2)^{0.5} - \tau_1 - \tau_2. \end{aligned}$$

The magnitude of a reversal of X_1 and X_2 relative to expectations is the same as the magnitude of a tie.

A close race between candidates M and M+1 can occur when their votes are within one of each other and all other candidates' votes are near their expected values. Thus, the magnitude of a close race involving M and M+1 is $-(\tau_M^{0.5} - \tau_{M+1}^{0.5})^2$.

Consider now some candidate $j > M+1$.

If candidates $1, \dots, M$ all got strictly more votes than $1+X_j$, then candidate j would not be in a close race.

Thus, when candidate j is in a close race, there at least one candidate i in $\{1, \dots, M\}$ such that $1+X_j \geq X_i$. The magnitude of this event is $-(\tau_i^{0.5} - \tau_j^{0.5})^2 \leq -(\tau_M^{0.5} - \tau_j^{0.5})^2$. So the magnitude of the event "j is in a close race" is not more than $-(\tau_M^{0.5} - \tau_j^{0.5})^2$. But if $\tau_j < \tau_{M+1}$ then this magnitude is strictly less than the magnitude of a close race between candidates M and M+1.

Thus, for any j in $\{M+2, \dots, K\}$, if $\tau_j < \tau_{M+1}$ then candidate j is not serious.

Consider now some candidate $i < M$.

If candidates $M+1, \dots, K$ all got strictly less votes than $X_i - 1$, then candidate i would not be in a close race, because he would be a guaranteed winner by at least 1 vote. So when candidate i is in a close race, there at least one candidate j in $\{M+1, \dots, K\}$ such that $1 + X_j \geq X_i$.

The magnitude of this event is $-(\tau_i^{0.5} - \tau_j^{0.5})^2 \leq -(\tau_i^{0.5} - \tau_{M+1}^{0.5})^2$.

So the magnitude of the event that j is in a close race is not more than $-(\tau_i^{0.5} - \tau_{M+1}^{0.5})^2$.

But if $\tau_i > \tau_M$ then this magnitude is strictly less than the magnitude of a close race between candidates M and $M+1$.

Thus, for any i in $\{1, \dots, M-1\}$, if $\tau_i > \tau_M$ then candidate i is not serious.

We obviously cannot have $\tau_i > \tau_M$, because then i would not be serious and so τ_i would be 0, contradicting $\tau_i > \tau_M \geq 0$. Thus we get the main result:

Theorem. For each i in $\{1, 2, \dots, M\}$, τ_i must be equal to τ_M .

For each j in $\{M+1, \dots, K\}$, τ_j must be equal to either τ_{M+1} or 0.

Incentives to cultivate favored minorities under alternative electoral systems by Roger Myerson, American Political Science Review (1993).

To win, should a politician appeal to all voters, or concentrate on special groups?
A model to show that the answer may depend on the electoral system:

Given K candidates in election to choose M winners ($K > M$, $M = 1 \dots$),
large number of voters (∞).

Winner gets a budget of \$1 per voter to allocate as promised in the campaign.

Each candidate i chooses a feasible offer distribution F_i on \mathbb{R}_+ with $\int_0^\infty x \, dF_i(x) = 1$.
Then each voter gets a promise from each candidate i independently drawn from F_i .
(If candidates were not independent, last-to-offer could win.)

Consider symmetric equilibria, where all candidates use same distribution F ,
and voters perceive any pair of candidates has same chance of being in close race.

Results for $K=2$, $M=1$, majority voting:

A feasible distribution F is an equilibrium iff

$$\forall G \text{ feasible } (\text{s.t. } \int_0^\infty x \, dG(x) = 1), \quad \int_0^\infty F(x) \, dG(x) \leq 1/2.$$

Unique equilibrium is Uniform $[0,2]$: $F(x) = x/2$ if $x \in [0,2]$, $F(x) = 1$ if $x > 2$.

$$\text{Then } \int_0^\infty F(x) \, dG(x) \leq \int_0^\infty x/2 \, dG(x) = [\int_0^\infty x \, dG(x)]/2 = 1/2.$$

Rank-scoring rules with K candidates.

A rank-scoring rule is characterized by $1=s_1 \geq s_2 \geq \dots \geq s_K=0$.

Each voter ranks the K candidates, gives s_1 to top-ranked, s_2 to second, s_j to the candidate ranked above $K-j$ others.

Candidate's score is his average points per voter. M high-scorers win (ties random).

Let \bar{S} denote the average points per voter, $\bar{S} = (s_1 + \dots + s_K)/K$.

Single-positive voting (plurality): $s_1=1, 0=s_2=\dots=s_K$. $\bar{S}=1/K$. [...*best-rewarding*]

Negative voting: $s_1=s_2=\dots=s_{K-1}=1, 0=s_K$. $\bar{S}=1-K/K$. [...*worst-punishing*]

V noncumulative votes: $s_1=\dots=s_V=1, 0=s_{V+1}=\dots=s_K$. $\bar{S}=V/K$.

Borda voting: $s_j = (K-j)/(K-1)$. $\bar{S}=1/2$.

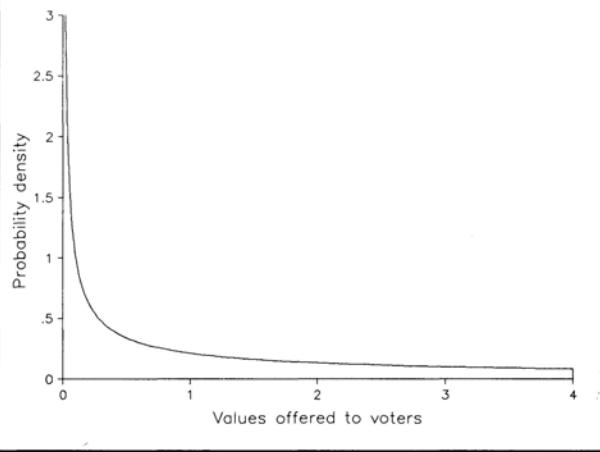
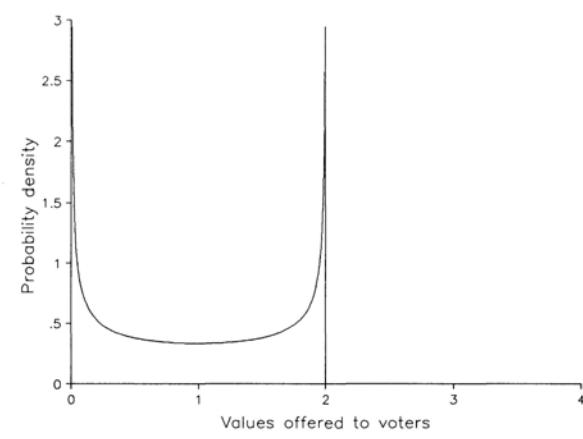
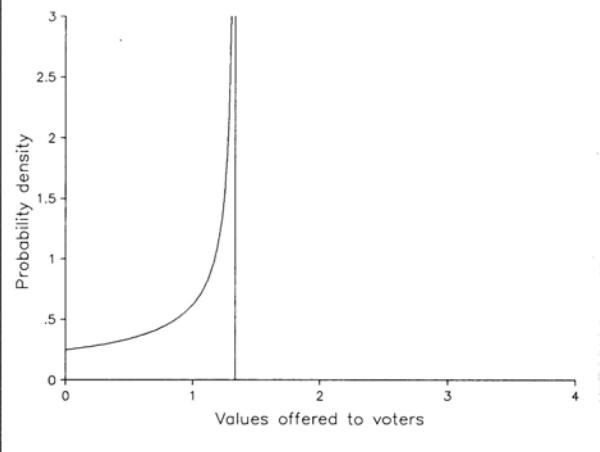
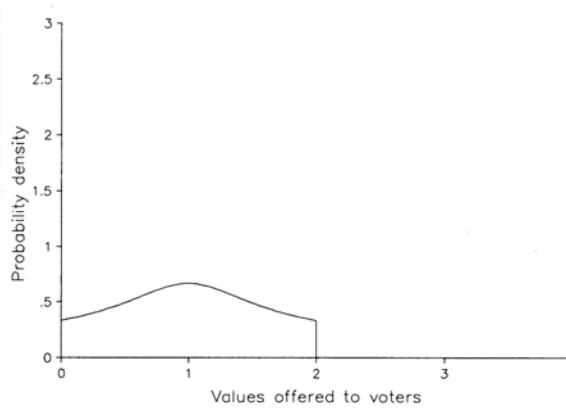
Let $R(p) = \sum_{j=1}^K s_j p^{K-j} (1-p)^{j-1} (K-1)! / [(j-1)!(K-j)!]$,

so $R(p)$ is the expected value of s_j when $K-j$ has a Binomial ($n=K-1, p$) distribution.
(Single-positive voting has $R(p) = p^{K-1}$.)

A feasible distribution F is an equilibrium iff $\forall G$ feasible,

$$\int_0^\infty R(F(x)) dG(x) \leq \int_0^\infty R(F(x)) dF(x) = \bar{S}.$$

Theorem 1. The unique symmetric equilibrium F has support $[0, 1/\bar{S}]$, and satisfies $x = R(F(x))/\bar{S}$ $\forall x \in [0, 1/\bar{S}]$, and so $F^{-1}(p) = R(p)/\bar{S}$ $\forall p \in [0, 1]$.

FIGURE 1**Offer Distribution with One Vote, Four Candidates****FIGURE 2****Offer Distribution with Two Votes, Four Candidates**
 $s_1=1, s_2=s_3=s_4=0. 1/\bar{S}=4.$
 $s_1=s_2=1, s_3=s_4=0. 1/\bar{S}=2.$
FIGURE 3**Offer Distribution with Three Votes, Four Candidates****FIGURE 4****Offer Distribution with Ranking Points (1, .5, .5, 0)**
 $s_1=s_2=s_3=1, s_4=0. 1/\bar{S}=4/3.$
 $s_1=1, s_2=s_3=0.5, s_4=0. 1/\bar{S}=2.$

Approval voting (a nonrank scoring rule):

In approval voting, each voter can give 0 or 1 point (approval) to each candidate. Winners have the M highest scores (most approvals).

Voter approves candidate i iff $(i's\ offer) > (\text{average of other candidates' offers})$.

When candidates use distribution F, average of m offers has cumulative A_m , where $A_1(x) = F(x)$, $A_m(x) = \int_0^{mx} A_{m-1}((mx-z)/(m-1)) dF(z)$.

When i promises x to a voter, this voter approves i with probability $A_{K-1}(x)$, and another candidate j is approved by this voter with probability $B(x)$, where $B(x) = \int_0^{\infty} A_{K-2}([(K-1)y-x]/[K-2]) dF(y)$.

The feasible distribution F is an equilibrium iff,
 $\forall G \text{ feasible}, \int_0^{\infty} A_{K-1}(x) dG(x) \leq \int_0^{\infty} B(x) dG(x)$.

Approval voting equilibria (discrete numerical approximation)

I found equilibria for discrete approximation, with offers being multiples of 0.05. The equilibrium offer distribution must have no atoms in continuous case, but the support seems to have many holes, mass in regions near 0 and in [1,2]. As K increases, fewer get offers near 0, maximal offer decreases toward 1.

FIGURE 5

Offer Distribution with Approval Voting,
Four Candidates

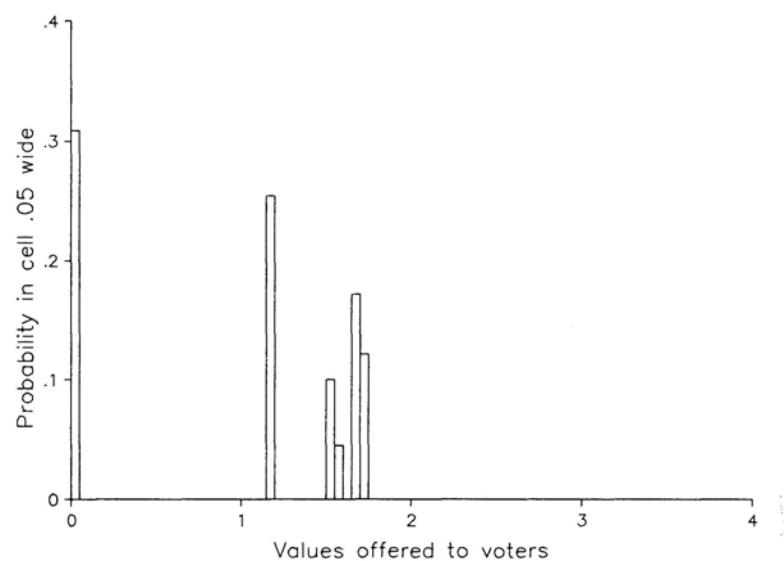


TABLE 2

Candidate Offers in Equilibrium under
Approval Voting

NUMBER OF CANDIDATES	S.D.	MAX. OFFER	FRACTION < 1	AVG. SCORE
2	.58	2.00	.50	.50
3	.71	1.90	.34	.55
4	.70	1.70	.31	.60
5	.66	1.60	.29	.63
6	.65	1.50	.29	.66
7	.61	1.45	.27	.69
8	.61	1.40	.27	.71
9	.56	1.35	.24	.74
10	.54	1.35	.23	.74

Note: Statistics were calculated using a .05 discrete approximation.

Single-transferable vote (STV)

In STV, each voter rank-orders the K candidates.

Candidates are eliminated one at a time in recounts, until M winners remain.

At each recount, each voter's ballot gives a point to his highest candidate among those who have not yet been eliminated; and then the lowest scorer is eliminated.

A feasible F is an equilibrium iff there does not exist any feasible G such that

$$\int_0^\infty (F(x))^{n-1} dG(x) \geq 1/n \quad \forall n \in \{M+1, \dots, K-1, K\}, \text{ with } > \text{ for some } n.$$

Theorem. For any $(\lambda_{M+1}, \dots, \lambda_{K-1}, \lambda_K)$ such that $\lambda_n > 0 \quad \forall n$, $\sum_{n=M+1}^K \lambda_n = 1$, there is an equilibrium offer distribution F with STV such that

$$\forall p \in [0,1], \quad F^{-1}(p) = \sum_{n=M+1}^K \lambda_n n p^{n-1}.$$

For any m in $\{M+1, \dots, K\}$, this F can approximate the equilibrium offer distribution for single-positive voting and m candidates, when $\lambda_m \approx 1$ and all other $\lambda_n \approx 0$.

Cox's threshold of diversity

A two-position model: Given K win-motivated candidates, each must simultaneously choose among two policy positions: Left and Right.

Assume that candidates at the same position are treated symmetrically by voters.

Fraction q of voters prefer Left, $1-q$ prefer Right.

Given K , Cox's threshold of diversity Q^* is the supremum of q such that there is a symmetric equilibrium in which all K candidates choose Right.

Fact. For any rank-scoring rule ($1=s_1 \geq s_2 \geq \dots \geq s_K = 0$), $Q^* = \bar{S} = (s_1 + \dots + s_K)/K$.

Single-positive voting yields $Q^* = 1/K$, and so small minority positions can win when K is large. (Best-rewarding yields low Q^* , diversity, favored minorities.)

Single-negative voting yields $Q^* = (K-1)/K$, so a majority can be neglected when K is large! (Ex: $K=10$ implies $Q^*=.9$; and with $q=.81$, we get $1-q=.19 > q/9=.09$)

(Worst-punishing yields high Q^* , clustering, making few enemies.)

Fact. Approval voting, Borda and STV yield $Q^* = 1/2$ for any K . (Majoritarian.)

A public-goods model: The winner of the election will get a budget of \$1 per voter to distribute as cash or to spend on a public good worth B to every voter.

An equilibrium where the public good is guaranteed exists only if $B \geq 1/Q^*$.

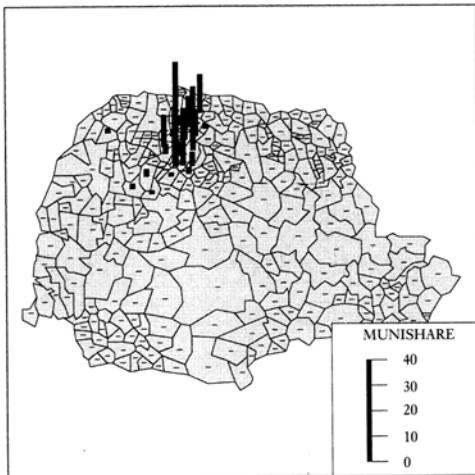
When $B < 1/Q^*$, if all other candidates promised the public good, then a candidate could win by promising $\$1/Q^*$ to a Q^* fraction of the voters.

Brazil uses open-list PR voting with single-positive voting for candidates. Figures from Barry Ames (1995):

FIGURE 1

THE CONCENTRATED-DOMINANT VOTE OF A LOCAL POLITICIAN

Municipal Vote Share of Said Ferreira, PMDB—Paraná

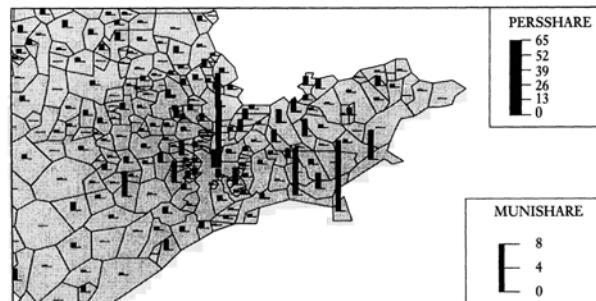


Parana: 30 seats.

FIGURE 2

THE CONCENTRATED-SHARED VOTE OF AN ENVIRONMENTALIST

Personal and Municipal Shares of Fábio Feldmann, PSDB—São Paulo



Sao Paulo: 60 seats.

FIGURE 3

THE SCATTERED-SHARED VOTE OF AN EVANGELICAL

Municipal Vote Share of Matheus Iensen, PTB—Paraná

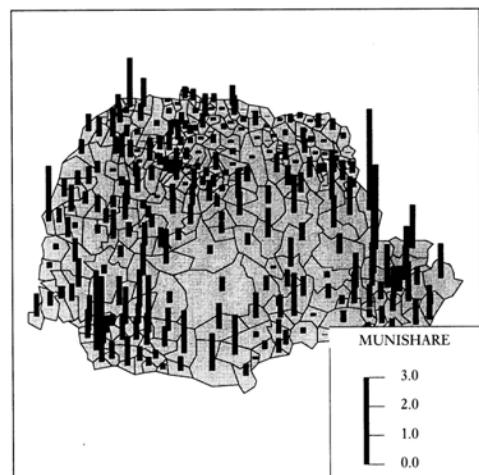
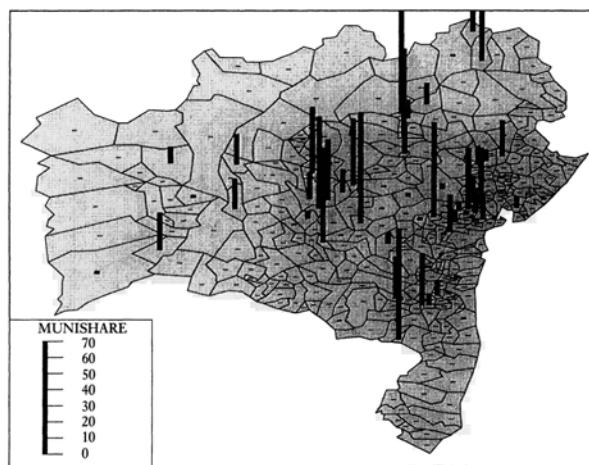


FIGURE 4

A SCATTERED-DOMINANT VOTE DISTRIBUTION

Municipal Vote Share of Jonival Lucas, PDC—Bahia



Bahia: 39 seats.

REFERENCES:

Roger Myerson, "Incentives to Cultivate Favored Minorities under Alternative Electoral Systems," *American Political Science Review* 87:856-869 (1993).

Barry Ames, "Electoral Rules, Constituency Pressures and Pork Barrel: Bases of Voting in the Brazilian Congress," *Journal of Politics* 57:324-343 (1995).

Barry Ames, *The Deadlock of Democracy in Brazil* (U Michigan Press, 2001).

Gary Cox, "Electoral equilibrium under alternative voting institutions," *American Journal of Political Science* 31:82-108 (1987).

Gary Cox, "Centripetal and centrifugal incentives in electoral systems," *American Journal of Political Science* 34:903-935 (1990).

Steven Reed, "Structure and Behaviour: Extending Duverger's Law to the Japanese Case," *British Journal of Political Science* 20:335-356 (1991).

Alessandro Lizzeri and Nicola Persico, "Provision of public goods under alternative electoral incentives," *American Economic Review* 91:225-245 (2001).

Alessandro Lizzeri and Nicola Persico, "A drawback of electoral competition," *Journal of the European Economic Association* 3(6):1318-1348 (2005).

Benoit Crutzen and Nicolas Sahugiet, "Redistributive politics and distortionary taxation," *Journal of Economic Theory* (2008).

Oliver Gross and R. A. Wagner, "A continuous Colonel Blotto game," *RAND Research Memorandum RM-408.*

Bipolar multicandidate elections with corruption

Myerson, Scandinavian Journal of Economics (2006) 108:727-742.

Set of candidates K is partitioned into $K_1 = \{\text{leftists}\}$ and $K_2 = \{\text{rightists}\}$.

Each candidate k has corruption level $f(k) \geq 0$.

k is clean if $f(k)=0$, corrupt if $f(k)>0$.

In game Γ_n , the number of voters is a Poisson random variable with mean n .

Each voter has a type t drawn independently from a probability distribution r that has a continuous positive density on the real line \mathbb{R} . $r(S) = \text{Prob}(\tilde{t} \in S) \quad \forall S \subseteq \mathbb{R}$.

A voter's type t measures his net preference for rightist candidates in K_2 ,
so t 's utility payoff if k wins is $u_k(t) = t - f(k)$ if $k \in K_2$, $u_k(t) = 0 - f(k)$ if $k \in K_1$.

Suppose $\forall i \in \{1, 2\}$, there exists a clean candidate k in K_i with $f(k)=0$. (wlog)

To complete the game, we must specify an electoral system (ties broken at random).

An equilibrium in the game Γ_n specifies a (weakly undominated) optimal strategy $\sigma_n(t)$ for each type t , and generates expected fractions $\tau_n(c)$ for each ballot c that is allowed in this electoral system, and win-probabilities $q_n(k)$ for each candidate k .

A large equilibrium (σ, τ, q) is a limit of (σ_n, τ_n, q_n) equilibria of Γ_n as $n \rightarrow \infty$.

A pair of candidates $\{i, j\}$ is distinct iff $u_i(t) \neq u_j(t)$ for some t in T .

$\{i, j\}$ -race is close when adding 1 vote could change winner from i to j , or j to i .

The $\{i, j\}$ -race is serious in a large equilibrium iff $\{i, j\}$ is a distinct pair and there is a strictly positive limit($n \rightarrow \infty$) of the conditional probability of a close $\{i, j\}$ -race given that some pair of distinct candidates are in a close race.

A candidate is serious iff he is involved in at least one serious race.

A candidate i is strong in a large eqm (σ, τ, q) iff $q(k) > 0$ (positive win-proby).

Theorem 1 (effectiveness against corruption). In a large equilibrium under approval voting, no corrupt candidates can be strong or serious.

Theorem 2 (majoritarianism). In a large equilibrium under approval voting, with probability 1, the winner will be a candidate who is considered best by at least half of the voters.

Failures of effective majoritarianism for other electoral systems (A):

In 3-candidate elections, consider rank-scoring rules where ballots are permutations of $(1, A, 0)$, for some A such that $0 \leq A \leq 1$.

Suppose $K_1 = \{1\}$, $K_2 = \{2, 3\}$, 1 and 2 are clean, 3 is corrupt.

If $A < 1/2$ then there is an equilibrium where $\{1, 3\}$ is the only serious race.

In this eqm, everybody votes $(1, A, 0)$ or $(0, A, 1)$, so winner will be either 1 or 3.

If $A \geq 1/2$ then 3 must be serious in all equilibria.

Otherwise, if 3 were not serious, then everybody would vote $(1, 0, A)$ or $(0, 1, A)$, but then 3 would always be in first place when 1 and 2 tie!

Failures of effective majoritarianism for other electoral systems (A,B):

Now consider scoring rules where ballots are permutations of $(1,A,0)$ and $(1,B,0)$, where $0 \leq A \leq B \leq 1$.

Approval voting is $(A,B)=(0,1)$, plurality voting is $(0,0)$,

Borda voting is $(1/2,1/2)$, negative voting is $(1,1)$.

Suppose now $K_1=\{1\}$, $K_2=\{2,3\}$, all three candidates are clean.

In a symmetric eqm, leftists randomize equally among $(1,A,0)$ and $(1,0,A)$, while rightists randomize equally among $(0,B,1)$ and $(0,1,B)$.

Notice $r < r(0+A)/2 + (1-r)(1+B)/2$ (1 loses) iff $r < (1+B)/(3+B-A)$.

$(1+B)/(3+B-A)$ is Cox's threshold of diversity here.

Also $1/2 < (1+B)/(3+B-A)$ iff $1 < A+B$.

When $1 < A+B$ and $1/2 < r(\mathbb{R}_-) < (1+B)/(3+B-A)$, then almost-surely leftists are a majority, but a rightist candidate wins (duplication helps rightists).

When $1 > A+B$ and $1/2 > r(\mathbb{R}_+) > (1+B)/(3+B-A)$ then almost-surely rightists are a majority, but the leftist candidate wins (duplication hurts rightists).

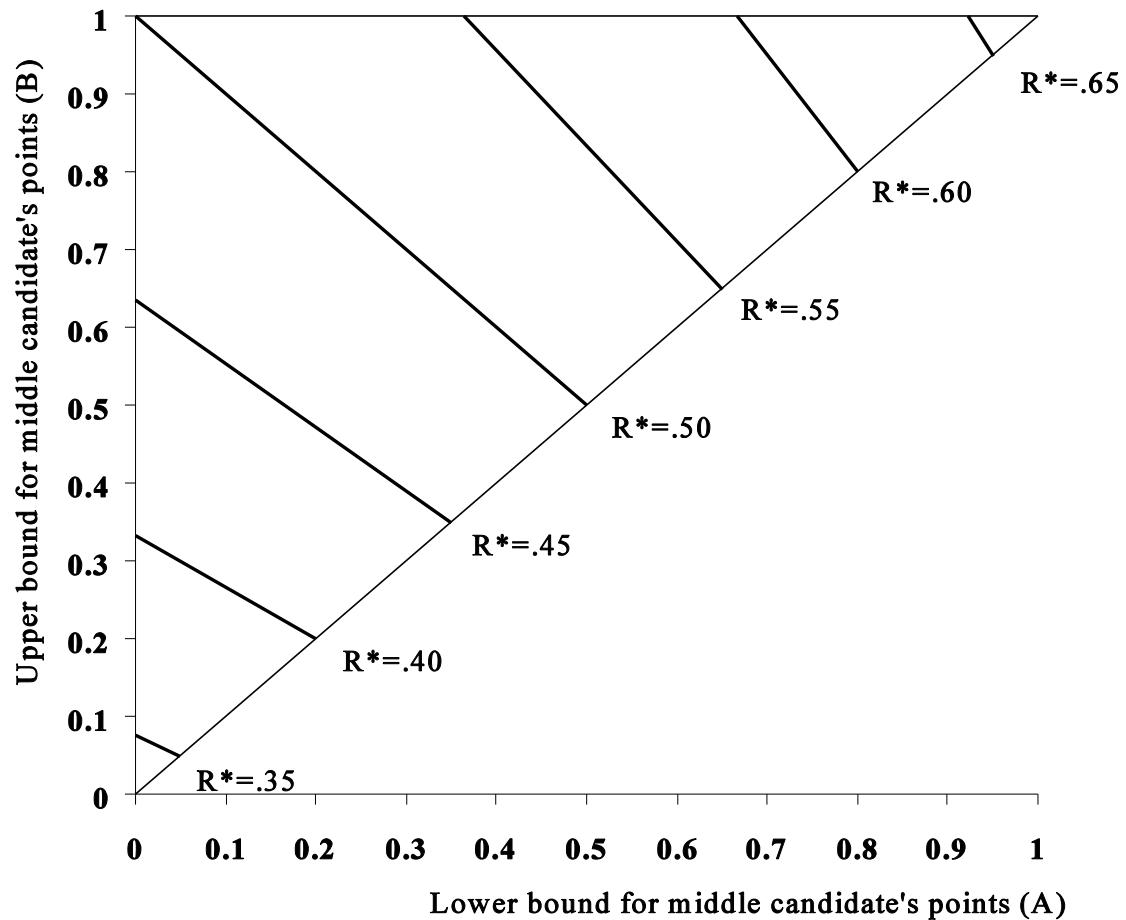


Figure 2. Cox's threshold of diversity (R^*) for (A,B)-scoring rules.

$R^* = (1+B)/(3+B-A) = [\text{biggest fraction that can lose with one candidate versus two in a symmetric eqm}]$
 R. Myerson, "Comparison of scoring rules in Poisson voting games," *JET* 103:219-251 (2002).

The magnitude of any event M is $\mu(M) = \lim_{n \rightarrow \infty} \log_e(P_n(M))/n$.

Lemma If $\{S_0, S_1, S_2, S_3\}$ is a partition of all voter-types, then the event "equal numbers of voters in S_1 and S_2 but no voters in S_0 " has magnitude $2\sqrt{r(S_1)r(S_2)} + r(S_3) - 1 = -(\sqrt{r(S_1)} - \sqrt{r(S_2)})^2 - r(S_0)$. If this event occurs, it is most likely that the numbers of voters in S_1 and S_2 are both very close to $n\sqrt{r(S_1)r(S_2)}$, the number of voters in S_3 is near $nr(S_3)$, and the number of voters in S_0 is (of course) 0.

Thm 1. In a large eqm under approval voting, corrupt candidates are not strong or serious.

Proof. All leftist voters with types in \mathbb{R}_- approve clean candidates in K_1 , as best among all candidates, because approving-best weakly dominates not-approving.

Similarly, all rightist voters in $\mathbb{R}_+ = [0, +\infty]$ approve clean candidates in K_2 .

If type t approves candidate i in K_1 and $s < t$ then type s also approves i

(because $u_i(s) - u_k(s) \geq u_i(t) - u_k(t) \quad \forall k \in K$, with " $=$ " if $k \in K_1$ and " $>$ " if $k \in K_2$).

So for i in K_1 , exists $\theta_n(i)$ such that t approves i in σ_n if $t < \theta_n(i)$ but not if $t > \theta_n(i)$.

For j in K_2 , exists $\theta_n(j)$ such that t approves j in σ_n if $t > \theta_n(j)$ but not if $t < \theta_n(j)$.

Let $\theta(k) = \lim_{n \rightarrow \infty} \theta_n(k)$.

Let $h_1 \in H_1 = \operatorname{argmax}_{i \in K_1} \theta(i)$, $h_2 \in H_2 = \operatorname{argmin}_{j \in K_2} \theta(j)$. (highest Escores on each side).

A clean candidate in K_1 has $\theta \geq 0$; a clean candidate in K_2 has $\theta \leq 0$. So $\theta(h_2) \leq 0 \leq \theta(h_1)$.

Let $r_1 = r([-\infty, \theta(h_2)])$, $r_2 = r([\theta(h_1), +\infty])$, $r_3 = r([\theta(h_2), \theta(h_1)])$.

The event of a close $\{h_1, h_2\}$ -race has magnitude $2\sqrt{r_1 r_2} + r_3 - 1$ (> -1).

Let i and j be any other candidates in K_1 and K_2 respectively.

Let $s_0 = r([\theta(h_2), \theta(j)] \cup [\theta(i), \theta(h_1)])$ (E fractn for- h_2 -but-not- j or for- h_1 -but-not- i),

$s_1 = r([-\infty, \min\{\theta(i), \theta(h_2)\}])$ (E fraction for- i -but-not- h_2),

$s_2 = r([\max\{\theta(j), \theta(h_1)\}, +\infty])$ (E fraction for- j -but-not- h_1),

$s_3 = r([\theta(j), \theta(i)])$ (E fraction for- i -and- j). Here $s_3 = 0$ if $\theta(j) \geq \theta(i)$.

The event of a close $\{i, j\}$ -race has magnitude $2\sqrt{s_1 s_2} + s_3 - 1$.

If $\theta(i) < \theta(h_1)$ or $\theta(h_2) < \theta(j)$ then $s_1 \leq r_1$, $s_2 \leq r_2$, $s_3 < r_3$, and

so a close $\{i, j\}$ -race has strictly lower magnitude than a close $\{h_1, h_2\}$ -race.

So a serious race between a leftist and rightist candidate can only involve candidates in H_1 and H_2 (those with highest expected scores on each side as $n \rightarrow \infty$).

Now suppose, contrary to the theorem, that some corrupt candidate is serious.

Let i denote the most corrupt serious candidate. Suppose w.l.o.g. that $i \in K_1$.

There must exist some j in H_2 such that the $\{i, j\}$ race is serious, because nobody would vote for i if i 's serious races were all with other less-corrupt candidates in K_1 .

Candidate i is the worst serious candidate for all voters in \mathbb{R}_+ , so $\theta_n(i) < 0 \ \forall n$.

Let g be a clean candidate in K_1 , who is approved by all voters in \mathbb{R}_- , so $\theta_n(g) \geq 0 \ \forall n$.

So the set of voters approving i is a subset of those approving g .

i can win only when all voters for- g -but-not-for- i vanish, leaving g in a tie with i .

So whenever an additional vote for i could make i win, there is a positive limiting conditional probability that the winner would be g otherwise.

But for type-0 voters, g is strictly better than i , and no serious candidate is worse than i .

So in the limit, there are strictly negative conditional expected gains for type-0 voters from approving i , given the event that some serious race is close.

So $\theta(i) < 0 \leq \theta(g)$. Thus, i is not in H_1 .

But then a close $\{i, j\}$ -race must have lower magnitude than some other close race involving a higher-expected-scoring candidate in H_1 .

So the $\{i, j\}$ race cannot be serious. This contradiction shows that no corrupt candidate i can be serious. Thus, all serious candidates must be clean.

A pair of clean candidates who are both in K_1 (or both in K_2) would not be distinct, so every serious race involves a clean candidate in K_1 and a clean candidate in K_2 .

In a one-winner election, strong candidates are serious, so all strong candidates are clean.

Thm 2. In a large equilibrium under approval voting, with probability 1, the winner will be a candidate who is considered best by at least half of the voters.

Proof. From Thm 1 all serious races are between clean candidates in K_1 and K_2 . So leftist voters in \mathbb{R}_- will all approve the clean candidates in K_1 but not in K_2 , while rightist voters in \mathbb{R}_+ will all approve the clean candidates in K_2 but not in K_1 . Corrupt candidates may get some votes, but only from an expected-strict subset of the voters on their same side of the political spectrum (so not serious contenders). So with probability 1, the winner will be a clean candidate from the side of the political spectrum that has a majority (or at least half) of the electorate, and so the winner will be an optimal candidate for at least half of the voters.

L. Bouton, M. Castanheira, "Divided majority and information aggregation."

Two states $\{\alpha, \beta\}$, $p(\alpha)=p(\beta)=0.5$. Three candidates $\{A, B, C\}$, three types $\{a, b, c\}$.

$$r(a|\alpha) = r(b|\beta) = .6 \times .6, \quad r(b|\alpha) = r(a|\beta) = .6 \times .4, \quad r(c|\alpha) = r(c|\beta) = 0.4.$$

$$\text{For } t \in \{a, b\}: \quad u_A(t|\alpha) = u_B(t|\beta) = 1, \quad u_B(t|\alpha) = u_A(t|\beta) = 0, \quad u_C(t|\alpha) = u_C(t|\beta) = -1.$$

$$\text{For } \omega \in \{\alpha, \beta\}. \quad u_C(c|\omega) = 1, \quad u_A(c|\omega) = u_B(c|\omega) = 0.$$

With plurality voting:

in one eqm, a's vote A, b's vote B, c's vote C, C almost-surely wins ($.4 > .36 > .24$);

in another eqm, a's and b's vote A, c's vote C, A almost-surely wins ($.6 > .4 > 0$);

in another eqm, a's and b's vote B, c's vote C, B almost-surely wins ($.6 > .4 > 0$).

With approval voting, consider symmetric scenarios where c's vote C,
 a's vote either AB with prob'y δ or A with prob'y $1-\delta$,
 b's vote either AB with prob'y δ or B with prob'y $1-\delta$.

Expected vote fractions:

$$\tau(C|\alpha) = 0.4, \quad \tau(A|\alpha) = 0.36(1-\delta), \quad \tau(AB|\alpha) = 0.6\delta, \quad \tau(B|\alpha) = 0.24(1-\delta),$$

$$\tau(C|\beta) = 0.4, \quad \tau(B|\beta) = 0.36(1-\delta), \quad \tau(AB|\beta) = 0.6\delta, \quad \tau(A|\beta) = 0.24(1-\delta).$$

In state α , magnitude of close AC race is $\mu(AC|\alpha) = -([.36(1-\delta)+.6\delta]^{0.5} - .4^{0.5})^2$,
 the magnitude of close BC race is smaller,

the magnitude of close AB race is $\mu(AB|\alpha) = -(1-\delta)(.36^{0.5} - .24^{0.5})^2$.

Magnitudes in state β are similar, just reversing the roles of A and B.

With $\delta=0.569$, get $\mu(AC|\alpha) = \mu(BC|\beta) = -0.0052 = \mu(AB|\alpha) = \mu(AB|\beta)$,
 and so we get an eqm with better-in- $\{A,B\}$ likely to win ($0.497 > 0.445 > 0.4$).

$(\delta > 0.569 \Rightarrow \mu(AC|\alpha) = \mu(BC|\beta) < \mu(AB|\alpha) = \mu(AB|\beta)$, so δ too large!)

$(\delta < 0.569 \Rightarrow \mu(AC|\alpha) = \mu(BC|\beta) > \mu(AB|\alpha) = \mu(AB|\beta)$, so δ too small!)

The **M+1 law** for multi-seat legislative elections under the single nontransferable vote (**SNTV**) system, was observed in Japanese legislative elections by S Reed in *Brit. J. Poli. Sci.* 20 (1990), formulated by G. Cox, *Amer. Poli. Sci. Rev.* 88 (1994).

Suppose that there are K candidates, numbered $1, 2, \dots, K$, in an election with single nontransferable vote where the top M candidates win. Here $M < K$.

Each voter has a type which is drawn independently from some finite set according to some fixed probability distribution r .

Each type t has a strict utility ranking $u_i(t)$ of candidates i .

A voter's payoff from the election is his sum of utility from each of the M winners.

In n 'th voting game, number of voters is a Poisson random variable with mean n .

Consider a large equilibrium, a convergent sequence of equilibria as $n \rightarrow \infty$.

The expected fraction who vote for each candidate i is converging to some limit τ_i .

Without loss of generality, we may number the candidates so that $\tau_1 \geq \tau_2 \geq \dots \geq \tau_K$.

Thm (weak M+1 law for SNTV). For each i in $\{1, 2, \dots, M\}$, τ_i must be equal to τ_M .

For each j in $\{M+1, \dots, K\}$, τ_j must be either τ_{M+1} or 0.

Thm. Consider a large eqm in SNTV with $\tau_1 \geq \tau_2 \geq \dots \geq \tau_K$. For each $i \leq M$, τ_i must equal τ_M . For each $j \geq M+1$, τ_j must be either τ_{M+1} or 0.

Lemma. The magnitude of a close race involving candidates M and $M+1$ is $-(\sqrt{\tau_M} - \sqrt{\tau_{M+1}})^2$.

Consider now some candidate $j > M+1$.

If candidates $1, \dots, M$ all got strictly more votes than $1 + \tilde{x}_j$, then candidate j would not be in a close race. Thus, when candidate j is in a close race, there at least one candidate i in $\{1, \dots, M\}$ such that $1 + \tilde{x}_j \geq \tilde{x}_i$. The magnitude of this event is $-(\sqrt{\tau_i} - \sqrt{\tau_j})^2 \leq -(\sqrt{\tau_M} - \sqrt{\tau_j})^2$. So the magnitude of the event that j is in a close race is not more than $-(\sqrt{\tau_M} - \sqrt{\tau_j})^2$.

But if $\tau_j < \tau_{M+1}$ then this magnitude is strictly less than the magnitude of a close race between candidates M and $M+1$. Thus we get

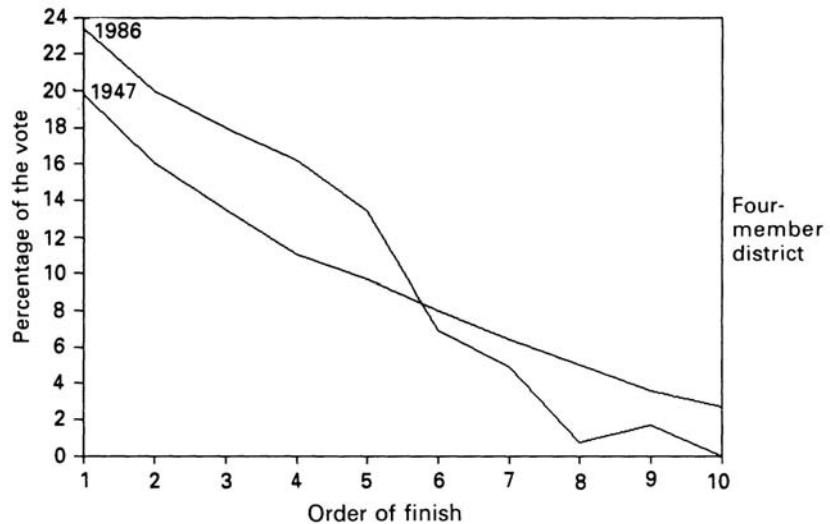
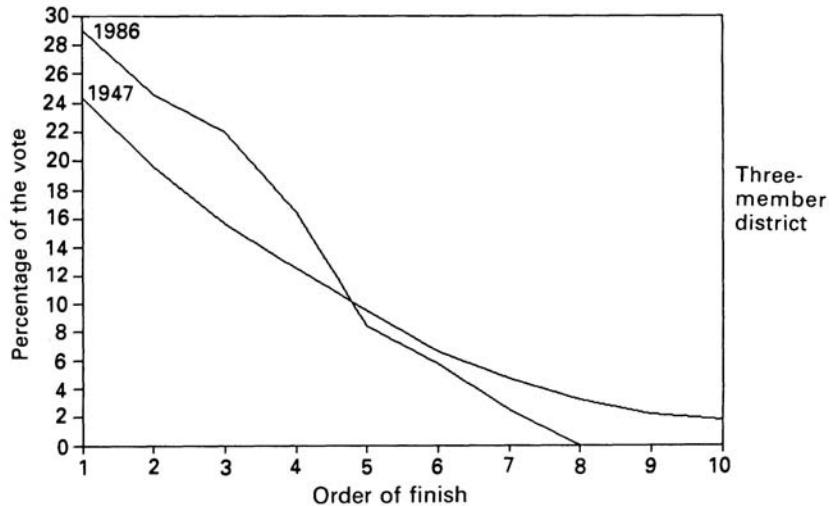
Lemma. For any j in $\{M+2, \dots, K\}$, if $\tau_j < \tau_{M+1}$ then candidate j is not serious.

Consider now some candidate $i < M$.

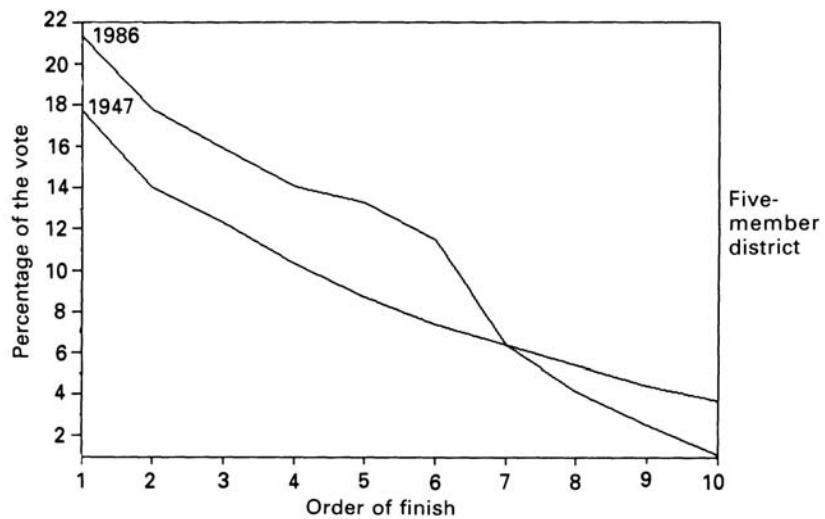
If candidates $M+1, \dots, K$ all got strictly less votes than $\tilde{x}_i - 1$, then candidate i would not be in a close race, because he would be a guaranteed winner even with one more vote. Thus, when candidate i is in a close race, there at least one candidate j in $\{M+1, \dots, K\}$ such that $1 + \tilde{x}_j \geq \tilde{x}_i$. The magnitude of this event is $-(\sqrt{\tau_i} - \sqrt{\tau_j})^2 \leq -(\sqrt{\tau_i} - \sqrt{\tau_{M+1}})^2$. So the magnitude of the event that j is in a close race is not more than $-(\sqrt{\tau_i} - \sqrt{\tau_{M+1}})^2$.

But if $\tau_i > \tau_M$ then this magnitude is strictly less than the magnitude of a close race between candidates M and $M+1$. Thus we get

Lemma. For any i in $\{1, \dots, M-1\}$, if $\tau_i > \tau_M$ then candidate i is not serious.



¹⁹ The first postwar election in 1946 was held under slightly different rules and in large electoral districts.



From: Steven R. Reed, "Structure and behavior: extending Duverger's law to the Japanese case," British J of Political Science 20(3), 340-341.

Fig. 1. Percentage of the vote by order of finish

Notes: Entries are the average votes of candidates who finished first, second, third and so on. The denominator is the number of candidates finishing in that order. When there are only five candidates in a district, the sixth place is not set to zero but ignored. Thus, it is possible for ninth-place finishers to have a higher average vote than eighth-place finishers, as in four-member districts, because most districts had only eight candidates.