Technology, Skill and Long Run Growth

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June 18, 2017 (Preliminary and incomplete)

Abstract

This paper develops a model in which two factors contribute to growth: investments in technology by heterogeneous firms and investments in human capital by heterogeneous workers. Growth in per capita income in turn takes two forms: growth in the quantity produced of each differentiated good and growth in the number of goods available. It is important to analyze both types of investment together because there is strategic complementarity in the incentives to invest. Workers invest in skill to increase their wages. But without continued improvement in the set of technologies used by firms, the returns to workers' investments would decline and, eventually, be too small to justify further investment. Similarly, without continued improvement in the skill distribution of the workforce, the incentives for firms to invest in better technologies would decline, and technology investment would eventually cease. Sustained growth requires continued investment in both factors.

Lionel McKenzie Lecture, SAET Meetings, Rio de Janeiro, 2017. I am grateful to Gadi Barlevy, Paco Buera, Jesse Perla, Ariel Burstein, Marios Angeletos, and participants the M&M Development Workshop at the Chicago Fed, the Minnesota Summer Workshop in Macroeconomics, and the IMF for useful comments.

1. OVERVIEW

This paper develops a model in which two factors contribute to growth: investments in technology by heterogeneous firms and investments in human capital by heterogeneous workers. Growth in per capita income in turn takes two forms: growth in the quantity produced of each differentiated good and growth in the number of goods available. The two forms will be referred to as total factor productivity (TFP) growth and growth in variety. Both types of investment affect both forms of growth, although the contributions are not symmetric.

It is important to analyze both types of investment together because there is strategic complementarity in the incentives to invest. Workers invest in skill to increase their wages. But without continued improvement in the set of technologies used by firms, the returns to workers' investments would decline and, eventually, be too small to justify further investment. Similarly, without continued improvement in the skill distribution of the workforce, the incentives for firms to invest in better technologies would decline, and technology investment would eventually cease. Sustained growth requires continued investment in both factors, and the contribution of this paper is to characterize the interplay between the two types of investment.

In the model here, the investment technologies for skill and technology are in many respects symmetric, and on balanced growth paths (BGPs) the rate of TFP growth is also the (common) growth rate of technology and human capital. Nevertheless, the actual growth rates of TFP and variety do not depend symmetrically on the parameters governing the investment processes for skill and for technology.

An improvement in any of the parameters governing the returns to investment in skill raises the rate of TFP growth and reduces the rate of growth in variety. An improvement in any of the parameters governing the returns to investment in technology raises the rate of variety growth, while the effect on TFP growth depends on preferences. In particular, the effect is positive, zero, or negative as the elasticity of intertemporal substitution (EIS) is greater than, equal to or less than unity, with the size of the effect depending on the magnitude of the preference for variety.

The production function for differentiated goods used here has two inputs, technology and human capital, and it is log-supermodular. Hence the competitive equilibrium features positively assortative matching between technology and skill. Both investment technologies have stochastic components, and the balanced growth path features stationary, nondegenerate distributions of technology and human capital, with both inputs growing at a common, constant rate.

The asymmetry in the two factors comes from the way entry appears. On the human capital side, growth in the size of the workforce is taken as exogenous, and both incumbent and entering workers engage in the same type of investment, to improve their existing skill or obtain initial skill.

On the technology side, entry is endogenous, governed by a zero-profit condition. Incumbent firms invest to improve their productivity—process innovation, and they die stochastically. Entering firms invest to obtain technologies for new goods—product innovation, and entrants face costs that incumbents do not. Hence the expected profitability of a new product guides the entry rate.

The rest of the paper is organized as follows. Related literature is discussed in section 2. Section 3 sets out the production technologies and characterizes the (static) production equilibrium, given the number of workers and producers and the distributions of skill and technology. In particular, it describes the allocation of labor—both quality and quantity—across technologies and the resulting prices, wages, output levels, and profits. Lemmas 1-3 establish the existence, uniqueness and efficiency of a production equilibrium, as well as some homogeneity properties. Proposition 4 shows that if the technology and skill distributions are Pareto, with locations that are appropriately aligned, then the equilibrium allocation of skill to technology is linear, and wage, price, and output and profit functions that are isoelastic.

Section 4 treats dynamics: the investment decisions of incumbent firms, new entrants, and workers; the evolution of the technology and skill distributions; and the interest rate and consumption growth. Section 5 provides formal definitions of a competitive equilibrium and a balanced growth path.

Section 6 specializes to the case where technology and skill have Pareto distributions, showing that the isoelastic forms for the profit and wage functions are inherited by the value functions for producers and workers. This fact leads to a tractable set of conditions describing investment and the evolution of the technology and skill distributions on a BGP. The first main result, Proposition 5, provides conditions that ensure the existence of a BGP.

Section 7 looks at the effects of various parameters on the growth rates of TFP and variety. Proposition 6, the second main result, describes these effects. Because the model has an important positive external effect, the competitive equilibrium investment rates are inefficient: they are too low. The effects of subsidies to investment by workers and firms are also studied.

Section 8 looks at some positive implications of the model: the wage dynamics for entering cohorts of workers and the revenue and employment dynamics for cohorts of entering firms. Section 9 concludes. Proofs and technical derivations and arguments are gathered in the Appendix.

2. RELATED LITERATURE

In most all of the endogenous growth literature, growth has only one source: either human capital accumulation or innovations in technology.

Among the human capital models, growth can arise from on-the-job learning, as in the learning-by-doing models of Arrow (1962), Stokey (1988), Young (1991, 1993), Matsuyama (1992). In others, human capital accumulation competes with production as a use of time, as in Uzawa (1965), Romer (1986), Lucas (1988, 2009), Lucas and Moll (2014), Perla and Tonetti (2014), and others.

In the literature on technology-driven growth, some models emphasize creative destruction, as in Romer (1990), Grossman and Helpman (1991), Aghion and Howitt (1992), Jones (1995), Stokey (1995), Acemoglu (2002), and Klette and Kortum (2004), while in others quality improvements are critical, as in Atkeson and Burstein (2010) and Luttmer (2007).

The model here is also related to the model of technology and wage inequality in Jovanovic (1998) and the model of skill and technology growth in Lloyed-Ellis and Roberts (2002).

The framework here builds on the model of technology growth across firms in Perla and Tonetti (JPE, 2014), adding a similar investment model on the human capital side.

3. PRODUCTION AND PRICES

The single final good is produced by competitive firms using intermediate goods as inputs. Intermediate goods are produced by heterogeneous, monopolistically competitive firms. Each intermediate firm produces a unique variety, and all intermediates enter symmetrically into final good production. But intermediate firms differ in their technology level x, which affects their productivity. Let N_p be the number (mass) of intermediate good producers, and let $F(x)$, with with continuous density f, denote the distribution function for technology.

Intermediate good producers use heterogeneous labor, differentiated by its human capital level h, as the only input. Let L_w, be the size of the workforce, and let $\Psi(h)$, with continuous density ψ, denote the distribution function for human capital. This section looks at the the allocation of labor across producers, and wages, prices, output levels, and profits, given N_p, F, L_w, Ψ.

A. Technologies

Although intermediates enter symmetrically into final good production, demands for them differ if their prices differ. Let $p(x)$ denote the price charged by a producer with technology x. The final goods sector takes these prices as given, and each final good producer has the CRS technology

$$y_F = \left[N_p^{1-\chi} \int y(x)^{(\rho-1)/\rho} f(x) dx \right]^{\rho/(\rho-1)}, \tag{1}$$

where $\rho > 1$ is the substitution elasticity and $\chi \in (0, 1/\rho]$ measures diminishing returns to increased variety.

Input demands are

$$y^d(x) = N_p^{-\rho\chi} \left(\frac{p(x)}{p_F} \right)^{-\rho} y_F, \qquad \text{all } x,$$

and prices will be normalized by setting the price of the final good to unity,

$$1 = p_F = \left[N_p^{1-\rho\chi} \int p(x)^{1-\rho} f(x) dx \right]^{1/(1-\rho)}. \tag{2}$$

The output of a firm depends on the size and quality of its workforce, as well as its technology. In particular, if a producer with technology x employs ℓ workers with human capital h, then its output is

$$y = \ell \phi(h, x),$$

where $\phi(h, x)$ is the CES function

$$\phi(h, x) \equiv \left[\omega h^{(\eta-1)/\eta} + (1-\omega)\, x^{(\eta-1)/\eta} \right]^{\eta/(\eta-1)}, \qquad \eta, \omega \in (0, 1). \tag{3}$$

The elasticity of substitution between technology and human capital is assumed to be less than unity, $\eta < 1$. Firms could employ workers with different human capital levels, and in this case their outputs would simply be summed. In equilibrium firms never choose to do so, however, and for simplicity the notation is not introduced.

B. Intermediate goods: price, output, labor

Let $w(h)$ denote the wage function. For a firm with technology x, the cost of producing one unit of output with labor of quality h is $w(h)/\phi(h, x)$. Optimal labor quality h^* minimizes this expression, so h^* satisfies

$$\frac{w'(h^*)}{w(h^*)} = \frac{\phi_h(h^*, x)}{\phi(h^*, x)}. \tag{4}$$

It is straightforward to show that if the (local, necessary) second order condition for cost minimization holds, then $\eta < 1$ implies h^* is strictly increasing. Unit cost

$$c(x) = \frac{w(h^*(x))}{\phi(h^*(x), x)},$$

is strictly decreasing in x,

$$\frac{c'(x)}{c(x)} = -\frac{\phi_x(h^*(x),x)}{\phi(h^*(x),x)} < 0,$$

where (4) implies that the other terms cancel.

As usual, profit maximization by intermediate good producers entails setting a price that is a markup of $\rho/(\rho-1)$ over unit cost. Output is then determined by demand, and labor input by the production function. Hence price, quantity, labor input, and operating profits for the intermediate firm are

$$\begin{aligned} p(x) &= \frac{\rho}{\rho-1}\frac{w(h^*(x))}{\phi(h^*(x),x)}, \\ y(x) &= N_p^{-\rho\chi}p(x)^{-\rho}y_F, \\ \ell(x) &= \frac{y(x)}{\phi(h(x),x)}, \\ \pi(x) &= \frac{1}{\rho}p(x)y(x), \qquad \text{all } x, \end{aligned} \tag{5}$$

where the price normalization requires (2). Firms with higher technology levels x have lower prices, higher sales, and higher profits. They may or may not employ more labor.

Each worker inelastically supplies one unit of labor. The labor market is competitive, and since the production function in (3) is log-supermodular, efficiency requires positively assortative matching (Costinot, 2009). Let x_m and h_m denote the lower bounds for the supports of F and Ψ. Then markets clear for all types of labor if

$$h_m = h^*(x_m), \tag{6}$$

$$L_w\left[1-\Psi(h^*(x))\right] = N_p\int_x^\infty \ell(\xi)f(\xi)d\xi, \qquad \text{all } x \geq x_m. \tag{7}$$

C. Production equilibrium

At any instant, the economy is described by its production parameters, the number of firms and workers, and the distributions of technology and skill.

DEFINITION: A *production environment* $\mathcal{E}_p$ is described by

i. parameters $(\rho, \chi, \omega, \eta)$, with $\rho > 1$, $\chi \in (0, 1/\rho]$, $\omega \in (0,1)$, $\eta \in (0,1)$;

ii. numbers of producers and workers $N_p > 0$ and $L_w > 0$;

iii. distribution functions $F(x)$ with continuous density $f(x)$ and lower bound x_m on its support, and $\Psi(h)$ with continuous density $\psi(h)$ and lower bound $h_m \geq 0$ on its support.

A production equilibrium consists of price functions and an allocation that satisfy profit maximization and labor market clearing.

DEFINITION: Given a production environment $\mathcal{E}_p$, the prices $w(h), p(x)$, and allocation $h^*(x)$, $y(x)$, $\ell(x)$, $\pi(x)$, y_F, are a *production equilibrium* if (2) and (4)-(7) hold.

The following result is then straightforward.

PROPOSITION 1: For any production environment $\mathcal{E}_p$, an equilibrium exists, and it is unique and efficient.

D. Homogeneity properties

The analysis of BGPs will exploit the fact that production equilibria have certain homogeneity properties. Lemma 2 deals with proportionate shifts in the two distribution functions.

LEMMA 2: Fix $\mathcal{E}_p$, and let $\mathcal{E}_{pA}$ be a production environment with the same parameters $(\rho, \chi, \omega, \eta)$ and numbers N_p, L_w, but with distribution functions F_A, Ψ_A satisfying

$$\begin{aligned} F_A(X) &= F(X/Q), \qquad \text{all } X, \\ \Psi_A(H) &= \Psi(H/Q), \qquad \text{all } H, \end{aligned}$$

where

$$Q \equiv \mathrm{E}_{F_A}(X).$$

If $[w, p, h^*, y, \ell, \pi, y_F]$ is the production equilibrium for $\mathcal{E}_p$, then the equilibrium for $\mathcal{E}_{pA}$ is

$$\begin{aligned} w_A(H) &= Qw(H/Q), & p_A(X) &= p(X/Q) \\ h_A^*(X) &= Qh^*(X/Q), & y_A(X) &= Qy(X/Q), \\ \ell_A(X) &= \ell(X/Q), & \pi_A(X) &= Q\pi(X/Q), \\ y_{FA} &= Qy_F, & &\text{all } X, H. \end{aligned}$$

Price and employment for any firm depend only on its relative technology $x = X/Q$, while its labor quality, output, and profits are scaled by Q. Wages and final output are also scaled by Q.

Lemma 3 deals with the effects of changes in the numbers of producers and workers. Define

$$\Omega \equiv \frac{1 - \rho\chi}{\rho - 1}. \tag{8}$$

and note that $\Omega \in [0, 1/(\rho - 1))$.

LEMMA 3: Fix $\mathcal{E}_p$, and let $\mathcal{E}_{pB}$ be a production environment with the same parameters and distribution functions, but with $L_{wB} = e^{\upsilon} L_w$ and $N_{pB} = e^n N_p$. If $[w, p, h^*, y, \ell, \pi, y_F]$ is the production equilibrium for $\mathcal{E}_p$, then the equilibrium for $\mathcal{E}_{pB}$ is

$$\begin{aligned} w_B &= e^{\Omega n} w, & p_B &= e^{\Omega n} p, \\ h_B^* &= h^*, & y_B &= e^{\upsilon - n} y, \\ \ell_B &= e^{\upsilon - n} \ell, & \pi_B &= e^{\upsilon + (\Omega - 1)n} \pi, \\ y_{FB} &= e^{\upsilon + \Omega n} y_F, & &\text{all } X, H. \end{aligned}$$

A change in L_w leads to proportionate changes in employment, output and profits at each firm and in final output, with wages, prices and the allocation of skill to technology unaffected.

An increase in N_p leads to proportionate decreases in employment and output at each firm. Final output, the price of each intermediate, and all wage rates change with an elasticity of $\Omega \geq 0$. Thus, all increase if variety is valued, if $\Omega > 0$, and all are unchanged if it is not, if $\Omega = 0$.

Profits per firm—which reflect both the increase in price and decrease in scale—can change in either direction. If $\Omega > 1$, then the love of variety is strong enough so that an increase in the number of producers actually increases the profit of each incumbent. This case occurs only if $\rho < 2$ and, in addition, the parameter χ is not too large. In the analysis of BGPs we will impose the restriction $\rho \geq 2$, to rule out this case.

E. Pareto distributions

In this section we will show that if the distribution functions F and Ψ are Pareto, with shape parameters that are not too different and location parameters that are appropriately aligned, the production equilibrium has a linear assignment of skill to technology, and wage, price, and profit functions that are isoelastic.

Proposition 4: Let $\mathcal{E}_p$ be a production environment for which F and Ψ are Pareto distributions with parameters (α_x, x_m) and (α_h, h_m). Assume that $\alpha_x > 1$, $\alpha_h > 1$, and

$$-1 < \alpha_x - \alpha_h < \rho - 1. \tag{9}$$

Define

$$\varepsilon \equiv \frac{1}{\rho}(1 + \alpha_x - \alpha_h) \in (0, 1), \tag{10}$$

$$a_h \equiv \left(\frac{1-\varepsilon}{\varepsilon} \frac{1-\omega}{\omega} \right)^{\eta/(\eta-1)}, \tag{11}$$

and in addition, assume

$$h_m = a_h x_m. \tag{12}$$

Then the production equilibrium for $\mathcal{E}_p$ has price and allocation functions

$$\begin{aligned}
h^*(x) &= a_h x, && \text{all } x, && (13)\\
w(h) &= N_p^{\Omega} w_0 x_m \left(h/x_m\right)^{1-\varepsilon}, && \text{all } h, && (14)\\
p(x) &= N_p^{\Omega} p_0 \left(x/x_m\right)^{-\varepsilon}, && \text{all } x, && (15)\\
y(x) &= L_w N_p^{-1} \frac{\alpha_h}{\alpha_x} \phi(a_h, 1) x_m \left(x/x_m\right)^{\rho\varepsilon}, && \text{all } x,\\
\ell(x) &= L_w N_p^{-1} \frac{\alpha_h}{\alpha_x} \left(x/x_m\right)^{\rho\varepsilon-1}, && \text{all } x,\\
\pi(x) &= L_w N_p^{\Omega-1} \pi_0 x_m \left(x/x_m\right)^{(\rho-1)\varepsilon}, && \text{all } x,\\
y_F &= L_w N_p^{\Omega} \frac{\alpha_h}{\alpha_x} \phi(a_h, 1) p_0^{\rho} x_m,
\end{aligned}$$

where

$$\begin{aligned}
p_0 &\equiv \left(\frac{\alpha_x}{\alpha_h - 1 + \varepsilon}\right)^{1/(\rho-1)},\\
w_0 &\equiv \frac{\rho-1}{\rho} p_0 \phi(a_h, 1) \frac{1}{a_h^{1-\varepsilon}},\\
\pi_0 &\equiv \frac{1}{\rho} p_0 \frac{\alpha_h}{\alpha_x} \phi(a_h, 1).
\end{aligned}$$

The shape parameters α_x and α_h need not be the same, although that is allowed, but (9) puts a restriction on how different they can be. The isoelastic forms for the wage and profit functions, together with the Pareto forms for the skill and technology distributions, will be important for analyzing BGPs.

4. DYNAMICS

In this section investment decisions and other dynamic aspects of the model are described. As in Perla and Tonetti (2014), investment is imitative, and for incumbent firms and workers it is a zero-one decision, with only an opportunity cost. The Pareto shape for the technology and skill distributions this investment technology requires for balanced growth fits well with the production environment here.

In the dynamic model, the set of firms/individuals at any date consists of producers/workers, investors and entrants. All firms/individuals exit exogenously at the fixed rates $\delta_x, \delta_h > 0$, and the technology/skill of producers/workers grow at the fixed rates μ_x, μ_h. The investment technology is symmetric for producers and workers. Briefly, it is as follows. A producer/worker can at any time abandon its current technology/skill and attempt to acquire a new one. Call this process innovation/retooling. The only cost of process innovation/retooling is an opportunity cost: the firm/worker cannot produce while investing. Success is stochastic, with fixed hazard rates $\lambda_{xi}, \lambda_{hi}$. Conditional on success, the process innovator/retooler receives a technology/skill that is a random draw from those of current producers/workers. Hence producers/workers switch to investing if and only if their technology/skill lies below an endogenously determined threshold X_m/H_m.

There is an important asymmetry between entering firms and entrants to the labor force, however. The labor force grows at a fixed rate υ, and entrants to the labor force pay no investment cost. They have a hazard rate for success λ_{he}, and are otherwise similar to retoolers.

Entering firms pay a one-time (sunk) cost, and a free entry condition determines the rate at which new firms enter. After paying the sunk cost they have a hazard rate for success λ_{xe}, and otherwise are similar to process innovators. Call entering firms product innovators.

Time is continuous and the horizon is infinite, $t \geq 0$. $N(t)$ is the total number of firms at date t, and $N_p(t), N_i(t), N_e(t)$ are the numbers of producers, process innovators, and product innovators; $L(t)$ is the total labor force at date t, and $L_w(t), L_i(t), L_e(t)$ are the numbers of workers, retoolers, and entrants; $F(X,t)$, $\Psi(H,t)$ are the distribution functions for technology and skill among producers and workers; $W(H,t)$, $H^*(X,t)$, $P(X,t)$, $Y(X,t)$, $\mathfrak{L}(X,t)$, $\Pi(X,t)$, $Y_F(t)$, $t \geq 0$, are the wage function, skill allocation, and so on; and $r(t)$ is the interest rate. Note that only producers and workers are identified by a technology or skill level, so the distribution functions describe only active producers and workers.

A. Firms: process and product innovation

Let $V^f(X,t)$ denote the value of a producer with technology X at date t. A firm that chooses to invest—a process innovator, stops producing, abandons its current technology, and waits to acquire a new one. The only cost of process innovation is the opportunity cost of the forgone profits. A process innovator cannot later reclaim its old technology, so all process innovators at date t are in the same position. Let $V_{fi}(t)$ denote their (common) value. All firms exit exogenously at the constant rate δ_x.

Success for process innovators is stochastic, arriving at rate λ_{xi}. Conditional on success at date t, the innovator gets a new technology that is random draw from the distribution $F(\cdot,t)$ among current producers. Hence $V_{fi}(t)$ satisfies the Bellman equation

$$[r(t) + \delta_x]\, V_{fi}(t) = \lambda_{xi} \left\{ \mathrm{E}_{F(\cdot,t)}[V^f(X,t)] - V_{fi}(t) \right\} + V'_{fi}(t), \qquad \text{all } t,$$

where the term in braces is the expected gain in value conditional on success.

The value $V^f(X,t)$ of a producer is the expected discounted value of its future profit flows. Clearly V^f is nondecreasing in its first argument: a better technology can only raise the firm's value. Hence at any date t, producers with technologies below some threshold $X_m(t)$ become process innovators, while those with technologies above the threshold continue to produce. It follows that at date t, the value of a producer with technology X is $V_{fi}(t)$ if $X \leq X_m(t)$, and the irreversibility of investment means that $X_m(t)$ is nondecreasing. While a firm produces, its technology X grows (or declines) at a constant rate μ_x. Hence the value $V^f(X,t)$ of a producer, a firm with $X > X_m(t)$, satisfies the Bellman equation[1]

$$[r(t) + \delta_x]\, V^f(X,t) = \Pi\,(X,t) + \mu_x X V_X^f(X,t) + V_t^f(X,t), \qquad \text{all } t.$$

Value matching provides a boundary condition for this ODE, and the optimal choice about when to invest implies that smooth pasting holds. Hence

$$\begin{aligned} V^f[X_m(t),t] &= V_{fi}(t), \\ V_X^f[X_m(t),t] &= 0, \qquad\qquad \text{all } t. \end{aligned}$$

Entering firms—product innovators—have a similar investment technology, except that they make a one-time (sunk) investment i_e, scaled by the average profits of current producers. The success rate of an innovator depends on his own investment i_e relative to the average spending rate $\bar{i}_e$ of others in his cohort, scaled by the ratio of entrants to existing products. In particular, let $E(t)$ denote the flow of new entrants

[1] At this stage, it would be easy to assume that the technology X of an incumbent evolves as a geometric Brownian motion. The Bellman equation would then become a (second-order) HJB equation, with the variance appearing as the coefficient on V_{XX}^f. The cross-sectional distribution of technologies among initially identical firms, within each age cohort, would be lognormal, with a growing variance, and the overall distribution would be a mixture of lognormals. When the solution to the model is actually characterized in section 6, however, the argument relies on technologies across incumbents having a Pareto distribution. At that point the mixture of lognormals would be incompatible with the requirement of a Pareto distribution overall, and the variance term would have to be dropped. Therefore, to streamline the notation and analysis, it is not introduced.

at t. The success rate of an entrant who invests i_e is $\Lambda_{xe} = \Phi_e(i_e/\bar{i}_e)/\left[E/\left(N_p + N_i\right)\right]$, where

$$\Phi_e(z) = \begin{cases} 0, & \text{if } z < 1 - \varepsilon_z, \\ \phi_e\left[1 - \left(1 - z\right)/\varepsilon_z\right], & \text{if } z \in \left[1 - \varepsilon_z, 1\right], \\ \phi_e\left[1 + \varepsilon_z\left(z - 1\right)\right], & \text{if } z > 1, \end{cases}$$

where $\phi_e > 0$ and where $\varepsilon_z > 0$ is small. Thus, the hazard function is very steep for z just below unity and very flat to the right. The form for Φ_e reflects a "patent race," with intense competition among potential entrants, and normalizing by $E/\left(N_p + N_i\right)$ reflects the reduced chances for success when the field is crowded.

Given Λ_{xe}, the value $V_{fe}(\cdot;\Lambda_{xe})$ of a product innovator who enters at t, gross of the investment cost, satisfies the Bellman equation

$$\left[r(\tau) + \delta_x\right] V_{fe}(\tau;\Lambda_{xe}) = \Lambda_{xe}\left[\mathrm{E}_{F(\cdot,\tau)}\left[V^f(X,\tau)\right] - V_{fe}(\tau;\Lambda_{xe})\right] + \frac{\partial V_{fe}(\tau;\Lambda_{xe})}{\partial \tau}, \qquad \tau \geq t.$$

Taking $\bar{i}_e(t)$ and $E(t)/N_p(t)$ as given, an entrant at t chooses its investment i_e to solve

$$\max_{i_e}\left\{V_{fe}\left[t;\frac{\Phi_e(i_e/\bar{i}_e)}{E/N_p}\right] - i_e\mathrm{E}_{F(\cdot,\tau)}\left[\Pi(X,t)\right]\right\}.$$

Since Λ_{xe} diverges as $E \rightarrow 0$, in equilibrium there is strictly positive entry at all dates. All entering firms choose the same investment level, and their (common) success rate is $\Lambda_{xe} = \phi_e/\left(E/N_p\right)$. Their (common) expenditure level is bid up to exhaust profits,

$$V_{fe}(t;\Lambda_{xe}) - i_e(t)\mathrm{E}_{F(\cdot,t)}\left[\Pi(X,t)\right] = 0, \qquad \text{all } t.$$

Let $I_E(t) = E(t)i_e(t)\ \mathrm{E}_{F(\cdot,t)}\left[\Pi(X,t)\right]$ denote aggregate spending by entrants at t.

B. Workers: investment in human capital

Each individual inelastically supplies one unit of labor to market activities, work or human capital accumulation, and his investment decisions maximize the expected discounted value of his lifetime earnings. All individuals die at the fixed rate $\delta_h > 0$.

As for firms, the investment decision of an individual is a zero-one choice, and the only cost is the opportunity cost of not working. An individual who chooses to invest—a retooler—stops working, abandons his old skill and waits to acquire a new one. Let $V^w(H,t)$ denote the value of a worker with skill H at date t, and let $V_{wi}(t)$ denote the value of a retooler.

Success for retoolers is stochastic, arriving at rate $\lambda_{hi} > 0$. Conditional on success, the individual gets a skill level drawn from the distribution $\Psi(\cdot, t)$ across the current workforce. Hence $V_{wi}(t)$ satisfies the Bellman equation

$$[r(t) + \delta_h] V_{wi}(t) = \lambda_{hi} \left\{ \mathrm{E}_{\Psi(\cdot,t)}[V^w(H,t)] - V_{wi}(t) \right\} + V'_{wi}(t), \qquad \text{all } t,$$

where the term in braces is the expected gain in value conditional on success.

Clearly $V^w(H,t)$ is nondecreasing in its first argument: higher human capital can only raise the worker's expected lifetime income. Hence at any date t, all individuals with skill below some threshold $H_m(t)$ become retoolers, while those with skill above the threshold continue working. It follows that at date t, the value of a worker with skill $H \le H_m(t)$ is $V_{wi}(t)$, and the irreversibility of investment implies that $H_m(t)$ is nondecreasing.

While a worker is employed, his human capital H grows (or declines) at a constant rate μ_h, which can be interpreted as on-the-job learning. Hence the value $V^w(H,t)$ for a worker with skill $H > H_m(t)$, satisfies the Bellman equation

$$[r(t) + \delta_h] V^w(H,t) = W(H,t) + \mu_h H V^w_H(H,t) + V^w_t(H,t), \qquad \text{all } t.$$

As for firms, value matching and smooth pasting hold at the threshold $H_m(t)$, so

$$\begin{aligned} V^w[H_m(t),t] &= V_{wi}(t), \\ V^w_H[H_m(t),t] &= 0, \qquad\qquad \text{all } t. \end{aligned}$$

New entrants into the workforce have an investment technology like the one for retoolers, except that their hazard rate for success, call it λ_{he}, may be different.They have no investment costs, so their value function $V_{we}(t)$ satisfies the Bellman equation

$$[r(t) + \delta_h]\, V_{we}(t) = \lambda_{he} \left\{ \mathrm{E}_{\Psi(\cdot,t)}[V^w(H,t)] - V_{wi}(t) \right\} + V'_{we}(t), \qquad \text{all } t.$$

The arrival at rate at any date, $\upsilon N(t)$ is proportional to population.

C. Flows of firms, the evolution of technology

Next consider the evolution of $N_p, N_i,\ N_e$, and the distribution function F. The number of producers $N_p(t)$ grows because of success by innovators of both types, and declines because of exit and decisions to switch to process innovation. The producers that switch to innovating around date t are those with technologies $X(t)$ that are close enough to the threshold $X_m(t)$ so that growth in that threshold overtakes them. Since technologies for producers grow at the rate μ_x, there is a positive level of switching at date t if and only if

$$X'_m(t) - \mu_x X_m(t) > 0, \qquad \text{all } t, \tag{16}$$

and

$$\begin{aligned} N'_p(t) \quad &= \quad \lambda_{xi} N_i(t) + \lambda_{xe} N_e(t) - \delta_x N_p(t) \\ &\qquad - \max\left\{0,\ [X'_m(t) - \mu_x X_m(t)]\, f[X_m(t), t] N_p(t)\right\}, \qquad \text{all } t. \end{aligned}$$

The number of process innovators N_i grows because producers switch to innovating, while the number of product innovators N_e grows because new entrants join. Each declines because of exit and success, so

$$\begin{aligned} N'_i(t) \quad &= \quad \max\left\{0,\ [X'_m(t) - \mu_x X_m(t)]\, f(X_m(t), t) N_p(t)\right\} - (\delta_x + \lambda_{xi})\, N_i(t), \\ N'_e(t) \quad &= \quad E(t) - (\delta_x + \lambda_{xe})\, N_e(t), \qquad \text{all } t. \end{aligned}$$

As a check, sum the three laws of motion to find that the total number of firms N grows because of entry and declines because of exit.

The distribution function F for technology among producers evolves because their technologies grow at rate μ_x, they exit at the rate δ_x, innovators of both types succeed, and firms at the threshold $X_m(t)$ switch to process innovations. As shown in the Appendix, $F(X,t)$ satisfies

$$-F_t(X,t) = f(X,t)\mu_x X + [1 - F(X,t)] \left[-\frac{N_p'(t)}{N_p(t)} - \delta_x + \lambda_{xi}\frac{N_i(t)}{N_p(t)} + \lambda_{xe}\frac{N_e(t)}{N_p(t)} \right],$$
$$\text{all } X \geq X_m(t), \quad t \geq 0.$$

D. Worker flows, the evolution of skill

Similarly, the number of workers L_w grows because of success by retoolers and entrants, and declines due to exit and decisions to switch to retooling. Workers who switch to retooling around date t are those whose human capital $H(t)$ falls below the (moving) threshold $H_m(t)$, despite growth at the rate μ_h. Hence workers are switching at date t if and only if

$$H_m'(t) - \mu_h H_m(t) > 0, \qquad \text{all } t, \tag{17}$$

and

$$\begin{aligned} L_w'(t) &= \lambda_{hi}L_i(t) + \lambda_{he}L_e(t) - \delta_h L_w(t) \\ &\quad - \max\left\{0, \ [H_m'(t) - \mu_h H_m(t)]\, \psi[H_m(t),t] L_w(t)\right\}, \qquad \text{all } t. \end{aligned}$$

The number of retoolers L_i grows because workers switch to to retooling, while the number of entrants $L_e(t)$ grows as individuals enter the labor force. Each declines because of exit and success, so.

$$\begin{aligned} L_i'(t) &= \max\left\{0, \ [H_m'(t) - \mu_h H_m(t)\psi[H_m(t),t]L_w(t)]\right\} - (\delta_h + \lambda_{hi})\, L_i(t), \\ L_e'(t) &= (\upsilon + \delta_h)\, L(t) - (\delta_h + \lambda_{he})\, L_e(t), \qquad \text{all } t. \end{aligned}$$

As a check, total population $L(t)$ grows at rate υ.

The distribution function $\Psi(H,t)$ for skill among workers evolves because their skills grow at rate μ_h, they exit at the rate δ_h, investors succeed at rate λ_h, and workers at the threshold H_m switch to retooling. Hence the law of motion for skill Ψ is like the one for F,

$$-\Psi_t(H,t) = \psi(H,t)\mu_h H + [1-\Psi(H,t)]\left[-\frac{L_w'(t)}{L_w(t)} - \delta_h + \lambda_{hi}\frac{L_i(t)}{L_w(t)} + \lambda_{he}\frac{L_e(t)}{L_w(t)}\right],$$
$$\text{all } X \geq X_m(t), \quad t \geq 0$$

E. Consumption

Individuals are organized into a continuum of identical, infinitely lived households of total mass one, where each dynastic household comprises a representative cross-section of the population. Newborns enter at the fixed rate $\delta_h + \upsilon$, so each household grows in size at the constant rate $\upsilon \geq 0$, and total population at date t is $L(t) = L_0 e^{\upsilon t}$.

Members of a household pool their earnings, so they face no consumption risk. The investment decisions of firms and workers, both incumbents and entrants, maximize, respectively, the expected discounted value of net profits and wages, discounted at market interest rates. Hence no further investment decisions are required at the household level.

The household's income consists of the wages of its workers plus the profits from its portfolio, which sum to output of the final good,

$$Y_F(t) = L_w(t)\mathrm{E}_{\Psi(\cdot,t)}\left[W(H,t)\right] + N_p(t)\mathrm{E}_{F(\cdot,t)}\left[\Pi(X,t)\right], \qquad \text{all } t.$$

That income is used for consumption and to finance the investment (entry) costs of new firms. Total investment costs for entrants are

$$I_E(t) = E(t)i_e(t)\mathrm{E}_{F(\cdot,t)}\left[\Pi(X,t)\right], \qquad \text{all } t,$$

and the household's net income at date t is $Y_F(t) - I_E(t)$.

All household members share equally in consumption, and the household has the constant-elasticity preferences

$$U = \int_0^\infty L_0 e^{\upsilon t} e^{-\hat{r}t} \frac{1}{1-\theta} c(t)^{1-\theta} dt,$$

where $\hat{r} > 0$ is the rate of pure time preference, $1/\theta > 0$ is the elasticity of intertemporal substitution, and $c(t)$ is per capita consumption.

Given interest rates $r(s)$, $s \geq 0$, define the cumulative interest factor

$$R(t) = \int_0^t r(s) ds, \qquad \text{all } t.$$

The household's consumption/savings decision, given interest rates and its net income stream, is to choose per capita consumption $c(t)$, $t \geq 0$, to maximize utility, subject to a lifetime budget constraint,

$$\int_0^\infty e^{-R(t)} \left\{ L_0 e^{\upsilon t} c(t) - [Y_F(t) - I_E(t)] \right\} dt \leq 0.$$

The condition for an optimum implies that per capita consumption grows at the rate

$$\frac{c'(t)}{c(t)} = \frac{1}{\theta} \left[r(t) - \hat{r} \right], \qquad \text{all } t,$$

with $c(0)$ determined by budget balance.

Final output is used for consumption and for the investment costs of entering firms. Hence market clearing for goods requires

$$Y_F(t) = L_0 e^{\upsilon t} c(t) + I_E(t), \qquad \text{all } t.$$

5. COMPETITIVE EQUILIBRIA, BGPS

This section provides formal definitions of a competitive equilibrium and a BGP. We start with the definition of a (dynamic) economy.

Definition: An *economy* $\mathcal{E}$ is described by

i. parameters $(\rho, \chi, \omega, \eta, \theta, \hat{r}, \upsilon)$, with $\rho > 1$, $\chi \in (0, 1/\rho)$, $\omega \in (0, 1)$, $\eta \in (0, 1)$, $\theta > 0$, $\hat{r} > 0$, $\upsilon \geq 0$;

ii. parameters $\delta_j, \lambda_{ji} > 0$ and μ_j, $j = h, x$, and $\lambda_{he} > 0$;

iii. an investment function Φ_e;

iv. initial conditions $N_{p0}, N_{i0}, N_{e0} > 0$, $L_{w0}, L_{i0}, L_{e0} > 0$;

v. initial distribution functions $F_0(X)$ with continuous density $f_0(X)$ and lower bound X_{m0} on its support, and $\Psi_0(H)$ with continuous density $\psi_0(H)$ and lower bound $H_{m0} \geq 0$ on its support.

A. Competitive equilibrium

The definition of a competitive equilibrium is standard.

DEFINITION: A *competitive equilibrium* of an economy $\mathcal{E}$ consists of the following, for all $t \geq 0$:

a. the numbers of producers, process innovators, product innovators, workers, and investing individuals, $[N_p(t), N_i(t), N_e(t), L_w(t), L_i(t),\ L_e(t)]$; and the inflow $E(t)$ of product innovators;

b. distribution functions $F(X;t), \Psi(H;t)$;

c. prices and allocations $[W(H;t), P(X,t), H^*(X;t), Y(X,t), \mathfrak{L}(X,t), \Pi(X,t), Y_F(t)]$;

d. value functions $\left[V^f(X,t),\ V_{fi}(t), V_{fe}(t)\right]$ for firms in each category, an investment threshold $X_m(t)$ for process innovators, an investment level $i_e(t)$ and success rate $\Lambda_{xe}(t)$ for product innovators;

e. value functions $[V^w(H;t), V_{wi}(t), V_{we}(t)]$, for individuals in each category, and an investment threshold $H_m(t)$ for retoolers;

f. aggregate investment costs $I_E(t)$, per capita consumption $c(t)$, and interest rate $r(t)$;

such that for all $t \geq 0$,

i. $[W, P, H^*, Y, \mathfrak{L}, \Pi, Y_F]$, is a production equilibrium, given $[N_p, L_w, F, \Psi]$;

ii. $\left(V^f, X_m\right)$ solve the investment problem of producers, given $[r, \Pi, V_{fi}]$; the success rate for product innovators is $\Lambda_{xe}(t) = \Phi_e(1)/\left[E/\left(N_p + N_i\right)\right]$; i_e satisfies the entry condition, and $[V_{fi}, V_{fe}]$ are consistent with $\left[r, V^f, F, \Lambda_{xe}\right]$;

iii. (V^w, H_m) solve the investment problem of workers, given r, W, V_{wi}; and $[V_{wi}, V_{we}]$ are consistent with $[r, V^w, \Psi]$;

iii. $[N_p, N_i, N_e, F]$ are consistent with $[X_m, E]$, and the initial conditions $[N_{p0}, N_{i0}, N_{e0}, F_0]$; and $X_m(0) = X_{m0}$;

v. $[L_w, L_i, L_e, \Psi]$ are consistent with H_m and the initial conditions $[L_{w0}, L_{i0}, L_{e0}, \Psi_0]$; and $H_m(0) = H_{m0}$;

vi. the investment cost I_E is consistent with i_e, E, Π, F; and c solves the consumption/savings problem of households, given $[r, L, Y_F, I_E]$; and

vii. the goods market clears.

B. Balanced growth

The rest of the analysis focuses on balanced growth paths, competitive equilibria with the property that quantities grow at constant rates, and the normalized distributions of technology and skill are time invariant.

Let $Q(t) \equiv E_{F(\cdot,t)}(X)$, $t \geq 0$, denote average technology at date t. On a BGP Q grows at a constant rate, call it g, and the distributions of relative technology $x = X/Q(t)$ and relative human capital $h = H/Q(t)$ are constant. By assumption total population L grows at the fixed rate υ, on a BGP the number of firms N also grows at a constant rate, call it n, and the shares of firms and individuals in each category, $[N_p/N,\ N_i/N, N_e/N]$ and $[L_w/L,\ L_i/L,\ L_e/L]$ are constant. The growth rates g and n are endogenous.

It follows from Lemma 2 that on a BGP the labor allocation in terms of relative technology and relative skill is time invariant. The growth rates for wages, prices, output levels, and so on are then described by Lemma 3. In particular, average product price grows at rate Ωn, where Ω is as in (8), average output per firm at rate $g + \upsilon - n$, and average employment per firm at rate $\upsilon - n$. Aggregate output, g_Y, the average wage, g_w, and the average profit per firm, g_π, grow at rates

$$\begin{aligned} g_Y &= g + \Omega n + \upsilon, \\ g_w &= g + \Omega n \qquad\qquad = g_Y - \upsilon, \\ g_\pi &= g + (\Omega - 1)\, n + \upsilon = g_Y - n, \end{aligned} \tag{18}$$

Per capita consumption grows at rate g_w. Aggregate consumption grows at rate g_Y, as do total investment costs $Ei_e \mathrm{E}_F[\Pi]$. If $\Omega > 1$, then love of variety is strong enough so that an increase in the number of producers actually raises the profits of each incumbent. In a dynamic model with free entry, this feature poses obvious problems. In the rest of the analysis we will assume that $\rho \geq 2$, which implies $\Omega < 1$.

These observations lead to the following definition.

Definition: A competitive equilibrium for $\mathcal{E}$ is a *balanced growth path* (BGP) if for some $g > 0$ and n, with $g_Y,\ g_w$ and g_π as in (18), the equilibrium has the property that for all $t \geq 0$:

a. the numbers of firms and individuals satisfy

$$\begin{aligned} N_p(t) &= e^{nt} N_{p0}, \qquad N_i(t) = e^{nt} N_{i0}, \qquad N_e(t) = e^{nt} N_{e0}, \\ L_w(t) &= e^{\upsilon t} L_{w0}, \qquad L_i(t) = e^{\upsilon t} L_{i0} \qquad L_e(t) = e^{\upsilon t} L_{e0}; \end{aligned}$$

and for some $E_0 > 0$, the number of entrants satisfies

$$E(t) = e^{nt} E_0;$$

b. for $Q_0 \equiv \mathrm{E}_{F_0}[X]$, average technology satisfies

$$Q(t) \equiv \mathrm{E}_{F(\cdot,t)}[X] = e^{gt} Q_0,$$

and for some $\left[\hat{F}(x), \hat{\Psi}(h)\right]$, the distribution functions satisfy

$$\begin{aligned} F(X,t) &= \hat{F}(X/Q(t)), \qquad \text{all } X, \\ \Psi(H,t) &= \hat{\Psi}(H/Q(t)), \qquad \text{all } H; \end{aligned}$$

c. for some $[w, p, h^*, y, \ell, \pi, y_F]$, the production equilibria satisfy

$$\begin{aligned} W(H;t) &= e^{g_w t} Q_0 w(H/Q(t)), \qquad \text{all } H; \\ P(X,t) &= e^{\Omega n t} Q_0 p(X/Q(t)), \\ H^*(X;t) &= e^{gt} Q_0 h^*(X/Q(t)), \\ Y(X,t) &= e^{(g+\upsilon-n)t} Q_0 y(X/Q(t)), \\ \mathfrak{L}(X,t) &= e^{(\upsilon-n)t} \ell(X/Q(t)), \\ \Pi(X,t) &= e^{g_\pi t} Q_0 \pi(X/Q(t)), \qquad \text{all } X; \\ Y_F(t) &= e^{g_Y t} Q_0 y_F; \end{aligned}$$

d. for some $[v_{fp}(x), v_{fi}, v_{fe}, x_m]$, the value functions and optimal policies for firms satisfy

$$\begin{aligned} V^f(X,t) &= e^{g_\pi t} Q_0 v_{fp}(X/Q(t)), \qquad \text{all } X, \\ V_{fi}(t) &= e^{g_\pi t} Q_0 v_{fi}, \\ V_{fe}(t) &= e^{g_\pi t} Q_0 v_{fe}, \\ X_m(t) &= e^{gt} Q_0 x_m, \end{aligned}$$

and $\Lambda(t) = \lambda_{xe}$ and $i_e(t) = i_{e0}$ are constant;

e. for some $[v_w(h), v_{wi}, v_{we}, h_m]$, the values and optimal policy for workers satisfy

$$\begin{aligned} V^w(H;t) &= e^{g_w t} Q_0 v_w(H/Q(t)), \qquad \text{all } H, \\ V_{wi}(t) &= e^{g_w t} Q_0 v_{wi}, \\ V_{we}(t) &= e^{g_w t} Q_0 v_{we}, \\ H_m(t) &= e^{gt} Q_0 h_m; \end{aligned}$$

f. for i_{e0} in (d) and some c_0, aggregate investment costs and consumption satisfy

$$\begin{aligned} I_E(t) &= e^{g_Y t} Q_0 i_{e0}, \\ C(t) &= e^{g_Y t} L_0 Q_0 c_0; \end{aligned}$$

and the interest rate satisfies

$$r(t) = r \equiv \hat{r} + \theta g_w.$$

BGPs arise—if at all—only for initial conditions $[N_{p0}, N_{i0}, N_{e0}, L_{w0}, L_{i0}, L_{e0}, F_0(X), \Psi_0(H)]$ that satisfy certain restrictions. The rest of the analysis focusses on a class of economies for which BGPs exist, and studies the determinants of the growth rates (g, n).

6. CONDITIONS FOR BALANCED GROWTH

In this section we will show that if an economy $\mathcal{E}$ has initial distribution functions F_0 and Ψ_0 that are Pareto, with shape and location parameters that satisfy the requirements of Proposition 4, then the normalized value functions $v_{fp}(x)$ and $v_w(h)$ for producers and workers inherit the isoelastic forms of the normalized profit and wage functions, and simple closed form solutions can be found. Moreover, (g, n) and the values $[v_{fi}, v_{fe}, x_m, v_{wi}, v_{we}, x_m, c_0, r]$ can be solved for explicitly, as well as the required ratios $[E/N, N_i/N_p, N_e/N_{p,} L_i/L_w, L_e/L_w]$ for a BGP. The arguments are summarized in Proposition 5, which provides sufficient conditions for a existence and uniqueness of a BGP.

A. Production equilibrium

Suppose the initial distributions F_0 and Ψ_0 are Pareto, with parameters (α_x, X_{m0}) and (α_h, H_{m0}). Assume (9) holds, define ε and a_h by (10) and (11), and assume that

$H_{m0} = a_h X_{m0}$. Average technology under the initial distribution is

$$Q_0 \equiv \mathrm{E}_{F_0}[X] = \frac{\alpha_x}{\alpha_x - 1} X_{m0}. \tag{19}$$

Use Q_0 to define the normalized distribution functions

$$\begin{aligned} \hat{F}(X/Q_0) &\equiv F_0(X), \qquad \text{all } X \geq X_m, \\ \hat{\Psi}(H/Q_0) &\equiv \Psi_0(H), \qquad \text{all } H \geq H_m. \end{aligned} \tag{20}$$

By construction $\mathrm{E}_{\hat{F}}(x) = 1$, so the location parameters for $\hat{F}$ and $\hat{\Psi}$ are

$$x_m = \frac{X_{m0}}{Q_0} = \frac{\alpha_x - 1}{\alpha_x}, \qquad h_m \equiv \frac{H_{m0}}{Q_0} = a_h x_m. \tag{21}$$

Hence the hypotheses of Proposition 4 hold for $\hat{F}$, $\hat{\Psi}$, and given N_{p0}, L_{w0}, the production equilibrium $[w, p, h^*, y, \ell, \pi, y_F]$ for the normalized distributions is as in (13)-(15).

B. Firms, investment in technology

Given x_m and $\hat{F}$, the Bellman equations for the three types of firms, together with the optimization condition for producers and the entry condition, determine $v_{fp}(x)$, v_{fi}, v_{fe}, and $\bar{i}_e$, as functions of (g, n), and also provide one additional equation relating (g, n). For convenience, define

$$\zeta \equiv 1 - (\rho - 1)\,\varepsilon. \tag{22}$$

Note that ζ can have either sign, and that $1 - \zeta = \varepsilon\,(\rho - 1) \in (0, \rho - 1)$.

Consider the investment decision and value of a producer. As shown in the Appendix, if Π and V^f have the forms required for a BGP, and $\pi(x)$ has the isoelastic form in Proposition 4, then the normalized value function $v_{fp}(x)$ for a producer satisfies the Bellman equation

$$(r + \delta_x - g_\pi)\, v_{fp}(x) = \pi_1 x^{1-\zeta} + (\mu_x - g)\, x v'_{fp}(x), \qquad x \geq x_m,$$

where

$$\pi_1 \equiv \pi_0 x_m^{\zeta} L_{w0}/N_{p0}^{1-\Omega}.$$

A BGP requires positive process innovation, which in turn requires $g > \mu_x$, so that the investment threshold grows faster than producers' technologies. Suppose this condition holds.

The normalized Bellman equation is a first-order ODE. Optimization by producers implies that at the investment threshold two conditions hold: value matching and smooth pasting. Using value matching for the boundary condition of the ODE, the solution is

$$v_{fp}(x) = B_x \pi_1 x^{1-\zeta} + \left(v_{fi} - B_x \pi_1 x_m^{1-\zeta}\right)(x/x_m)^{R_x}, \qquad x \geq x_m, \tag{23}$$

where the constant $B_x > 0$ and characteristic root $R_x < 0$ are known. The first term in (23) is the value of a producer who operates forever, never investing. The second term represents the additional value from the option to invest in process innovation. The smooth pasting condition

$$v_{fi} = B_x \pi_1 x_m^{1-\zeta}\left(1 - \frac{1-\zeta}{R_x}\right), \tag{24}$$

determines v_{fi} as a function of π_1.

On a BGP the normalized value v_{fi} for a process innovator satisfies the Bellman equation

$$(r + \delta_x - g_\pi)\, v_{fi} = \lambda_{xi}\left\{\mathrm{E}_{\hat{F}}\left[v_{fp}(x)\right] - v_{fi}\right\}.$$

Substituting for $\mathrm{E}_{\hat{F}}\left[v_{fp}(x)\right]$ and v_{fi} from (23) and (24), and factoring out $\pi_1 B_x x_m^{1-\zeta}$, gives

$$r + \delta_x - g_\pi = \frac{-R_x(1-\zeta)\lambda_{xi}}{(\alpha_x - 1 + \zeta)(\alpha_x - R_x)}. \tag{25}$$

Recall that r, g_π and R_x involve (g, n), while all of the other parameters in this expression are exogenous. Hence (25) is one equation in the pair (g, n).

For an entrant, a product innovator, the normalized value v_{fe} satisfies the Bellman equation

$$(r+\delta_x-g_\pi+\lambda_{xe})\,v_{fe}(\lambda_{xe})=\lambda_{xe}\mathrm{E}_{\hat{F}}\left[v_{fp}(x)\right], \tag{26}$$

where the success rate λ_{xe} and investment i_{e0} are determined as follows.

Fix $\bar{i}_e$ and $\epsilon = E/\left(N_p+N_i\right)$. Then an entrant who invests less than $\left(1-\varepsilon_z\right)\bar{i}_e$ has no chance of success, and one that invests $i_e \geq \left(1-\varepsilon_z\right)\bar{i}_e$ has success rate

$$\lambda_{xe}=\frac{\phi_e}{\epsilon}\times\begin{cases}\left[1-\left(1-i_e/\bar{i}_e\right)/\varepsilon_z\right], & \text{if } i_e/\bar{i}_e\in\left[1-\varepsilon_z,1\right],\\ \left[1+\varepsilon_z\left(i_e/\bar{i}_e-1\right)\right], & \text{if } i_e/\bar{i}_e>1.\end{cases}$$

The entrant chooses its investment to maximize $v_{fe}(\lambda_{xe})-i_e\mathrm{E}_{\hat{F}}\left[\pi(x)\right]$. Conditional on entering, the unique optimal choice is $i_e=\bar{i}_e$ if and only if

$$\varepsilon_z\leq\frac{v'_{fe}(\phi_e/\epsilon)}{\mathrm{E}_{\hat{F}}\left[\pi(x)\right]}\frac{\phi_e}{\epsilon}\frac{1}{\bar{i}_e}\leq\frac{1}{\varepsilon_z},$$

and for this investment level $\lambda_{xe}=\phi_e/\epsilon$. Entering is preferred to staying out if and only if the value from doing so covers the sunk cost, and strictly positive profits would encourage more entry, an increase in E. Hence the entry holds with equality,

$$v_{fe}(\phi_e/\epsilon)=\bar{i}_e\mathrm{E}_{\hat{F}}\left[\pi(x)\right].$$

In this case, calculating $v'_{fe}(\lambda_{xe})$ from (26), and evaluating at $\lambda_{xe}=\phi_e/\epsilon$, we find that an optimum at $i_e=\bar{i}_e$ requires

$$\varepsilon_z\leq\frac{r+\delta_x-g_\pi}{r+\delta_x-g_\pi+\phi_e/\epsilon}\frac{\phi_e}{\epsilon}\leq\frac{1}{\varepsilon_z}, \tag{27}$$

which holds for $\varepsilon_z>0$ sufficiently small. The investment rate $i_{e0}=\bar{i}_e$ is determined by the entry condition,

$$\begin{aligned} i_{e0} &= \frac{v_{fe}(\lambda_{xe})}{\mathrm{E}_{\hat{F}}\left[\pi(x)\right]}\\ &= \frac{\lambda_{xe}}{r+\delta_x-g_\pi+\lambda_{xe}}\frac{\mathrm{E}_{\hat{F}}\left[v_{fp}(x)\right]}{\mathrm{E}_{\hat{F}}\left[\pi(x)\right]}. \end{aligned} \tag{28}$$

The success rate is $\lambda_{xe}=\phi_e/\epsilon$, where the entry flow rate ϵ is determined below.

C. Workers, investment in skill

The argument for workers is analogous except that entry is exogenous, as is the hazard rate λ_{he}. Hence the normalized value function v_w satisfies the Bellman equation

$$(r + \delta_h - g_w)\, v_w(h) = w_1 h^{1-\varepsilon} + (\mu_h - g)\, h v_w'(h),$$

where

$$w_1 \equiv w_0 h_m^{\varepsilon} N_{p0}^{\Omega}.$$

A BGP requires positive retooling, which requires $g > \mu_h$. Suppose this condition holds. Using value matching for the boundary condition, the solution to this ODE is

$$v_w(h) = B_h w_1 h^{1-\varepsilon} + \left(v_{wi} - B_h w_1 h_m^{1-\varepsilon}\right)(h/h_m)^{R_h}, \qquad h \geq h_m, \tag{29}$$

where the constant $B_h > 0$ and characteristic root $R_h < 0$ are known. The first term in (29) is the value of a worker who never invests, and the second represents the additional value from the option to retool. The smooth pasting condition

$$v_{wi} = B_h w_1 h_m^{1-\varepsilon}\left(1 - \frac{1-\varepsilon}{R_h}\right), \tag{30}$$

determines v_{wi}.

The value v_{wi} for a retooler satisfies the Bellman equation

$$(r + \delta_h - g_w)\, v_{wi} = \lambda_{hi}\left\{\mathrm{E}_{\Psi}\left[v_w(h)\right] - v_{wi}\right\}.$$

Using (29) and (30) to substitute for $\mathrm{E}_{\hat{\Psi}}[v_w]$ and v_{wi}, and factoring out $w_1 B_h h_m^{1-\varepsilon}$, gives

$$r + \delta_h - g_w = \frac{-R_h(1-\varepsilon)\lambda_{hi}}{(\alpha_h - 1 + \varepsilon)(\alpha_h - R_h)}, \tag{31}$$

a second equation in the pair (g, n).

The value v_{we} of an entrant to the workforce is determined by the Bellman equation

$$(r + \delta_h - g_\pi + \lambda_{he})\, v_{we} = \lambda_{he}\mathrm{E}_{\hat{\Psi}}\left[v_w(h)\right]. \tag{32}$$

D. Flows of firms and workers, the evolution of technology and skill

On a BGP the number of firms grows at a constant rate n. Hence the entry rate is the sum of the growth and exit rates,

$$E = (n + \delta_x) N. \tag{33}$$

The shares of firms engaged in production and the two kinds of innovation are constant over time. Given λ_{xe}, the laws of motion for N_p, N_i and N_e imply that the ratios of process and product innovators to producers are

$$\begin{aligned} \frac{N_i}{N_p} &= \frac{\alpha_x (g - \mu_x)}{n + \delta_x + \lambda_{xi}}, \\ \frac{N_e}{N_p} &= \frac{n + \delta_x}{\lambda_{xe}} \left[1 + \frac{\alpha_x (g - \mu_x)}{n + \delta_x + \lambda_{xi}} \right]. \end{aligned} \tag{34}$$

Similarly, the laws of motion for L_w, L_i, and L_e imply that the ratios of retoolers and those acquiring initial skill to workers are

$$\begin{aligned} \frac{L_i}{L_w} &= \frac{\alpha_h (g - \mu_h)}{\upsilon + \delta_h + \lambda_{hi}}, \\ \frac{L_e}{L_w} &= \frac{\upsilon + \delta_h}{\lambda_{he}} \left[1 + \frac{\alpha_h (g - \mu_h)}{\upsilon + \delta_h + \lambda_{hi}} \right]. \end{aligned} \tag{35}$$

It is easy to check that if X_m and H_m grow at rate g, as required on a BGP, then the distribution functions $F(\cdot, t)$ and $\Psi(\cdot, t)$ evolve as required.

The pair (ϵ, λ_{xe}) is determined as follows. By definition

$$\begin{aligned} \epsilon &= \frac{E}{N_p + N_i} = \frac{E}{N} \frac{N}{N_p + N_i} \\ &= (n + \delta_x) \left(\frac{n + \delta_x}{\lambda_{xe}} + 1 \right), \end{aligned}$$

where the second line uses the expressions above for the two ratios. On a BGP $\lambda_{xe} = \phi_e / \epsilon$, so ϵ satisfies

$$\epsilon = (n + \delta_x) \left[(n + \delta_x) \frac{\epsilon}{\phi_e} + 1 \right].$$

Hence

$$\epsilon = \frac{(n+\delta_x)\,\phi_e}{\phi_e - (n+\delta_x)^2},$$

and

$$\lambda_{xe} = \frac{\phi_e}{n+\delta_x} - (n+\delta_x)\,. \tag{36}$$

E. Consumption, the interest rate

On a BGP per capita consumption grows at the rate g_w, so the interest rate is

$$r = \hat{r} + \theta g_w. \tag{37}$$

Aggregate income grows at the rate g_Y, so its present discounted value is finite if and only if $r > g_Y$, or

$$\hat{r} > g_Y - \theta g_w = \upsilon + (1-\theta)\,(g + \Omega n)\,. \tag{38}$$

Using (28) for i_{e0}, market clearing for goods in the normalized production equilibrium requires

$$y_F = L_0 c_0 + i_{e0}, \tag{39}$$

which determines the initial level of per capita consumption, c_0.

F. Existence of a BGP

The growth rates (g, n) are determined by (25) and (31). Substituting for $g_w, g_\pi, r,$ and the roots R_x and R_h, those two equations are

$$\begin{aligned} g &= \frac{1}{\xi_x}\left[\upsilon - n + \alpha_x\mu_x + \frac{1-\zeta}{\alpha_x - 1 + \zeta}\lambda_{xi} - \delta_x - \hat{r} - (\theta-1)\,\Omega n\right], \\ g &= \frac{1}{\xi_h}\left[\alpha_h\mu_h + \frac{1-\varepsilon}{\alpha_h - 1 + \varepsilon}\lambda_{hi} - \delta_h - \hat{r} - (\theta-1)\,\Omega n\right], \end{aligned} \tag{40}$$

where

$$\xi_x \equiv \alpha_x - 1 + \theta > 0, \qquad \xi_h \equiv \alpha_h - 1 + \theta > 0,$$

and the signs follow from the fact that $\alpha_x, \alpha_h > 1$.

Propositions 5 and 6 both use the assumption $\rho \geq 2$, which implies $\Omega < 1$. One additional joint restriction on ρ, α_h is also imposed if $\theta < 1$. Although stronger than required for existence, it will be needed for the comparative statics results.

PROPOSITION 5: Let $\mathcal{E}$ be an economy with:

a. $\rho \geq 2$, and $\rho > \alpha_h / (\alpha_h - 1)$ if $\theta < 1$;

b. initial distributions F_0, Ψ_0 that are Pareto, with shape and location parameters (α_x, X_{m0}), (α_h, H_{m0}) satisfying the hypotheses of Proposition 4.

Define ε and ζ by (10) and (22). Then the pair of equations in (40) has a unique solution (g, n), and there are unique $[Q_0, \hat{F}, \hat{\Psi}]$, $[w, p, h^*, y, \ell, \pi, y_F]$ satisfying conditions (b)-(c) for a BGP; $[v_{fp}(x), v_{f0}, v_{fe}, x_m, \lambda_{xe}, i_{e0}]$ satisfying (d); $[v_w(h), v_{w0}, h_m]$ satisfying (e); and $[c_0, r]$ satisfying (f).

If in addition:

c. the initial ratios $[N_{i0}/N_{p0}, N_{e0}/N_{p0}]$ and $[L_{i0}/L_{w0}, L_{e0}/L_{w0}]$ satisfy (34) and (35); and

d. $g > \mu_x$, $g > \mu_h$, (27) holds, (38) holds and $c_0 > 0$,

then $\mathcal{E}$ has a unique competitive equilibrium that is a BGP.

PROOF: For existence and uniqueness of a solution to (40), it suffices to show that the two equations are not collinear. Here we will prove a slightly stronger result, that

$$\frac{1}{\xi_x} [1 + (\theta - 1) \Omega] > \frac{1}{\xi_h} (\theta - 1) \Omega,$$

or

$$\alpha_h > (\theta - 1) [(\alpha_x - \alpha_h) \Omega - 1] . \tag{41}$$

Since $\chi \in (0, 1/\rho]$ implies $\Omega \in [0, 1/(\rho - 1))$, and (9) implies $\alpha_x - \alpha_h \in (-1, \rho - 1)$, it follows that

$$(\alpha_x - \alpha_h) \Omega - 1 \in \left(-\frac{\rho}{\rho - 1},\ 0 \right).$$

If $\theta \geq 1$, the term on the right in (41) is zero or negative. If $\theta < 1$, then by assumption $\alpha_h > \rho/(\rho - 1)$. In either case (41) holds, and there exists a unique (g, n) satisfying (40).

Define $[Q_0, \hat{F}, \hat{\Psi}]$, x_m, h_m by (19)-(21). Since (21) implies (12) holds, by Proposition 4 the normalized production equilibrium $[w, p, h^*, y, \ell, \pi, y_F]$ is described by (13)-(15), so the price and allocation functions are isoelastic.

Then (23), (24) and (26) determine $[v_{fp}, v_{fi}, v_{fe}]$; (28) determines i_{e0}; (29), (30) and (32) determine $[v_w, v_{wi}, v_{we}]$; (33) determines E_0; (36) determines λ_{xe}; (37) and (39) determine r and c_0; and (d) implies that the growth rates satisfy the required inequalities and that $c_0 > 0$. Hence the solution describes a BGP. ■

7. GROWTH RATES ON THE BGP

In this section we will look at how various parameters affect the growth rates (g, n). We will assume throughout that the hypotheses of Proposition 5 hold. We will start with three special cases, where $\theta = 1$ or $\Omega = 0$ or $\alpha_x = \alpha_h$, and then look at the general case.

A. Special cases

Suppose preferences are logarithmic, $\theta = 1$. Then $\xi_h = \alpha_h$, and the second equation in (40) simplifies to

$$g = \mu_h + \frac{1}{\alpha_h}\left[\frac{1-\varepsilon}{\alpha_h - 1 + \varepsilon}\lambda_{hi} - \delta_h - \hat{r}\right].$$

In this case g is a weighted sum of the parameters $(\mu_h, \lambda_{hi}, \delta_h, \hat{r})$ governing skill accumulation and the rate of time preference, and the parameters $(\mu_x, \lambda_{xi}, \delta_x)$ governing technology accumulation do not enter. Faster human capital growth μ_h among on-the-job workers raises g, as does a higher success rate λ_{hi} for retoolers. A higher exit rate δ_h or a higher discount rate $\hat{r}$ reduces g.

The weights depend on the elasticity parameters α_h and $1-\varepsilon$. A higher value for $1-\varepsilon$ increases the elasticity of the wage with respect to skill, increasing the returns to investment and increasing g.

Recall that ε is in turn increasing in $1/\rho$ and α_x, and decreasing in α_h. Since $1/\rho$ measures the monopoly power of firms, an increase in monopoly power reduces g. An increase in α_x, which decreases the mean of the technology distribution and makes the Pareto tail thinner, also reduces g. An increase in α_h decreases the mean of the skill distribution, and the direct effect is to reduce g. The indirect effect, through ε, is in the reverse direction, but presumably smaller.

Next, suppose that variety is not valued, so $\chi = 1/\rho$ and $\Omega = 0$. In this case, too, the growth rate g is determined by the second equation in (40),

$$g = \frac{1}{\xi_h}\left[\alpha_h\mu_h + \frac{1-\varepsilon}{\alpha_h - 1 + \varepsilon}\lambda_{hi} - \delta_h - \hat{r}\right],$$

and the solution is qualitatively like the logarithmic case.

Finally, suppose that the two Pareto distributions have the same shape parameter, $\alpha_x = \alpha_h$. Then $\varepsilon = 1/\rho$ and $\zeta = \varepsilon$, and subtracting the second line in (40) from the first gives

$$n = \upsilon + \alpha_h\left(\mu_x - \mu_h\right) + \frac{\rho - 1}{\rho\left(\alpha_h - 1\right) + 1}\left(\lambda_{xi} - \lambda_{hi}\right) - \left(\delta_x - \delta_h\right).$$

Thus, the growth rate n of variety is equal to the rate of population growth υ, adjusted for differences in the exogenous rates of growth, success, and exit between firms and workers, $\mu_x - \mu_h$, $\lambda_{xi} - \lambda_{hi}$, and $\delta_x - \delta_h$. A higher rate of population growth, or a larger difference in the rates of growth or success, $\mu_x - \mu_h$ or $\lambda_{xi} - \lambda_{hi}$, increases the rate of variety growth, while a larger difference in the exit rates, $\delta_x - \delta_h$, decreases it.

B. General case

Proposition 6 extends these comparative statics results to the general case.

PROPOSITION 6: Let $\mathcal{E}$ be as in Proposition 5. Then

a. an increase in μ_h, λ_{hi} or a decrease in δ_h raises g and reduces n;

b. an increase in $\upsilon, \mu_x, \lambda_{xi}$ or a decrease in δ_x raises n, and

—raises g if $(\theta - 1)\,\Omega < 0$,

—has no effect on g if $(\theta - 1)\,\Omega = 0$, and

—reduces g if $(\theta - 1)\,\Omega > 0$;

c. a decrease in $\hat{r}$ raises g if $(\theta - 1)\,\Omega \leq 0$, and has otherwise ambiguous effects.

PROOF: First we will show that the equations in (40), plotted in n-$g-$space, are as shown Figure 1: the line defined by the first equation is downward sloping; the line defined by the second equation has a positive, zero, or negative slope as $(\theta - 1)\,\Omega < 0$, $= 0$, or > 0; and in all case the second line crosses the first from below.

For the first claim, note that $\rho \geq 2$ implies $\Omega \in [0, 1)$, so $[(\theta - 1)\,\Omega + 1] > 0$. The second claim is obvious, and the third follows from (41).

Then claims (a) - (c) follow directly from Figure 1. As shown in panel (a), an increase in μ_h or λ_{hi}, or a decrease in δ_h, shifts the second line upward, increasing g and decreasing n. As shown in panel (b), an increase in υ, μ_x or λ_{xi}, or a decrease in δ_x, shifts the first line to the right, increasing n. The effect on g depends on the slope of the second line. A decrease in $\hat{r}$ does both, as shown in panel (c). Hence it raises g if $(\theta - 1)\,\Omega \leq 0$, and otherwise the effects depend on the relative slopes of the two lines. ■

Changes in the initial population size and number firms, L_0 and N_0, do not affect the growth rates, although they do affect the levels for profits and wage rates.

C. Policies to increase growth

Competitive equilibria in the model here are inefficient. Investments by producers/workers have positive external effects, since they improve the pools of technologies/skills from which later investors draw. Since this positive externality is not taken into account by individual firms or workers, the competitive equilibrium has too little investment compared with the efficient level, as in Perla and Tonetti (2014, Propositions 3 and 4). Subsidies to investment are obvious policies to overcome this inefficiency.

A complete analysis of the optimal policies would require looking at the transition path between the old and new BGPS, and is beyond the scope of this paper. But it is easy to assess the long-run impact of a small subsidy to either type of investment on the rates of TFP growth and growth in variety.

Consider a subsidy to process innovators at a fixed rate σ_x, scaled by the average profits of current producers, and a subsidy to retoolers at a fixed rate σ_h, scaled by the average wage of current workers. Then the normalized Bellman equations for process innovators and retoolers are

$$\begin{aligned}(r+\delta_x)\,v_{f0} &= \sigma_x \mathrm{E}_F\left[\pi(x)\right] + \lambda_{xi}\left\{\mathrm{E}_F[v^f(x)] - v_{f0}\right\} + g_\pi v_{f0},\\ (r+\delta_h)\,v_{w0} &= \sigma_h \mathrm{E}_\Psi\left[w(h)\right] + \lambda_{hi}\left\{\mathrm{E}_\Psi[v^w(x)] - v_{w0}\right\} + g_w v_{w0},\end{aligned}$$

and the pair of equations in (40) becomes

$$\begin{aligned} g &= \frac{1}{\xi_x}\left[\upsilon - n + \alpha_x\mu_x + \frac{1-\zeta}{\alpha_x(1-\sigma_x)-1+\zeta}\lambda_{xi} - \delta_x - \hat{r} - (\theta-1)\,\Omega n\right],\\ g &= \frac{1}{\xi_h}\left[\qquad \alpha_h\mu_h + \frac{1-\varepsilon}{\alpha_h(1-\sigma_h)-1+\varepsilon}\lambda_{hi} - \delta_h - \hat{r} - (\theta-1)\,\Omega n\right].\end{aligned}$$

The subsidies increase the coefficients on the hazard rates λ_{xi} and λ_{hi}. Hence by Proposition 6 a subsidy $\sigma_h > 0$ to retoolers raises g and reduces n, while a subsidy $\sigma_x > 0$ to process innovators increases n and increases, leaves unchanged, or decreases g as $(\theta-1)\,\Omega < 0$, $= 0$, or > 0.

8. WAGE AND EMPLOYMENT/REVENUE DYNAMICS

This section looks at the empirical predictions of the model for wage growth for individuals and growth in revenue and employment for firms. Growth for either type of agent has two components, continuous growth while working/producing, and jumps from successful investment. Since the jumps are hard to match with data, we will focus on age cohorts of individuals and firms.

Each age cohort of individuals has a mix of workers, retoolers, and new entrants, with proportions that change as the cohort ages. Only workers receive wages. Over time the share of workers among survivors grows, as entrants succeed in acquiring skill. Call that share $\sigma_w(a)$. As shown in the Appendix,

$$\sigma_w(a) = \frac{\lambda_{hi}}{b_h} - e^{-b_{hi}a}\frac{\lambda_{he}}{b_h}\frac{b_h - \lambda_{hi}}{b_h - \lambda_{he}} + e^{-\lambda_{he}a}\frac{\lambda_{he} - \lambda_{hi}}{b_h - \lambda_{he}}, \qquad a \geq 0,$$

where $b_h \equiv \alpha_h\left(g - \mu_h\right) + \lambda_{hi}$. Thus, the share of workers in the cohort is zero at entry and grows monotonically as the cohort ages, converging to λ_{hi}/b_h.

The workers in any age cohort have average wages equal to the economy-wide average. Since average wages grow at the rate g_w, average earnings (across all individuals) in the cohort at age a is

$$e_{Av}(a) = e^{g_w a}\sigma_w(a)W_0, \qquad a \geq 0,$$

where W_0 is the average wage in the economy when the cohort entered. Hence average earnings among survivors grows monotonically as a function of age, at a rate that declines toward g_w in the long run.

The argument for firms is analogous. Each age cohort of firms has a mix of producers, process innovators, and product innovators, and only producers have revenue and employees. Call the share of producers $\sigma_p(a)$. Then

$$\sigma_p(a) = \frac{\lambda_{xi}}{b_x} - e^{-b_x a}\frac{\lambda_{xe}}{b_x}\frac{b_x - \lambda_{xi}}{b_x - \lambda_{xe}} - e^{-\lambda_{xe}a}\frac{\lambda_{xi} - \lambda_{xe}}{b_x - \lambda_{xe}}, \qquad a \geq 0,$$

where $b_x \equiv \alpha_x (g - \mu_x) + \lambda_{xi}$.

At any date average revenue (average employment) is the same among producers in any age cohort, growing at rate g_π (at rate $\upsilon - n$). Hence average revenue and average employment among all surviving firms in the cohort at age a are

$$\begin{aligned} R_{Av}(a) &= e^{g_\pi a}\sigma_p(a)R_0, \\ \bar{\ell}_{Av}(a) &= e^{(\upsilon-n)a}\sigma_p(a)\bar{\ell}_0, \qquad a \geq 0, \end{aligned}$$

where R_0 and $\bar{\ell}_0$ are average revenue and employment across all firms when the cohort entered. Hence the average revenue among survivors grows monotonically as a function of age, at a rate that declines toward g_π in the long run. Average employment among survivors grows when the cohort is young. It continues to grow in the long run if and only if population growth exceeds variety growth, if and only if $\upsilon - n > 0$. The absolute number of survivors in the cohort declines over time as firms exit, so cohort totals for revenue and employment are scaled by $e^{-\delta_x a}$.

9. CONCLUSION

The contribution of this paper is to develop a model in which both technological change and human capital accumulation are required to sustain long run growth. The main results are to provide conditions for the existence of a BGP, and to show how the rates of TFP and variety growth depend on various parameters of the model.

On a BGP, skill and technology grow at a common rate. Nevertheless, the parameters governing skill accumulation are more important than those governing technological change in determining that rate. The parameters for skill and technology enter more symmetrically—but with opposite signs—in determining growth in product variety. Thus improvements in the parameters for technological change encourage entry, while improvements in the parameters for skill accumulation encourage investment in both skill and technology, but discourage growth in variety.

The production function here is log supermodular, so in equilibrium there is positively assortative matching between technologies and skills. As a consequence, continued investment in either factor remains worthwhile only because the other grows. If investment in one factor were to cease for some reason, investment in the other would eventually cease as well.

Thus, there is no 'race' between technology and skill: they grow together. Although transitional dynamics are not studied here, the results suggest that if one factor started out with a distribution that was 'ahead' of the other, investment in that factor would slow down—and perhaps would cease altogether, while the incentive to invest in the lagging factor would be exceptionally strong. In this general sense, the analysis suggests that the system would converge to a BGP. Obviously, convergence is possible only if the initial distributions are themselves Pareto—or at least have Pareto tails, since the dynamics of investment do not change the shape of the tails.

In the model here, TFP growth comes from imitation of incumbents. Hence investments in skill and technology have positive external effects, improving the distribution offered to later investors. It follows that the competitive equilibrium is inefficient: investment is too low, for both factors, as in Perla and Tonetti (2014).

The model suggests a number of questions for further work. In terms of theory, the model could be used to analyze the effects of a change in the (exogenous) rate of population growth or the transition path for an economy that received an inflow of technology from outside. On the empirical side, the implications for wages and revenue, sketech in section 8, could be further developed, and one could ask if the the transition paths in rapidly developing countries look like the model's prediction for an economy where technology gets "ahead" because of inflows from abroad.

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APPENDIX A: PRODUCTION AND PRICES

A. Production equilibrium

Proof of Proposition 1: Use (5) to write labor demand as

$$\ell(x) = N_p^{-\rho\chi}\left(\frac{\rho-1}{\rho}\right)^{\rho}\frac{\phi(h^*(x),x)^{\rho-1}}{w(h^*(x))^{\rho}}y_F, \qquad \text{all } x \geq x_m, \tag{42}$$

and differentiate (7) to write labor market clearing as

$$\begin{aligned} L_w\psi(h^*(x))h^{*\prime}(x) &= N_p\ell(x)f(x), \qquad \text{all } x \geq x_m, && (43)\\ L_w &= N_p\int_{x_m}^{\infty}\ell(\xi)f(\xi)d\xi. && (44)\end{aligned}$$

For any fixed y_F, (4) and (43) are a pair of ODEs in $w(h)$ and $h^*(x)$, with $\ell(x)$ given by (42). The price normalization (2) is a boundary condition for w, and (6) is the boundary condition for h^*. Then (44) determines y_F, and (5) determines p, y, π. ■

Proof of Proposition 4: For the wage function in (14) and the CES function ϕ in (3), optimal labor quality in (4) is as in (13), with a_h as in (11). Then (5) simplifies to (15), where the constants w_0, p_0, π_0, involve only exogenous parameters. Hence by Lemma 1 it suffices to show that (2), (6), (43) and (44).

Using the fact that F is a Pareto distribution, the prices p in (15) satisfy (2) if

$$\begin{aligned} 1 &= N_p^{1-\rho\chi}N_p^{\Omega(1-\rho)}p_0^{1-\rho}x_m^{(1-\rho)\varepsilon}\alpha_x x_m^{\alpha_x}\int_{x_m}^{\infty}x^{(\rho-1)\varepsilon}x^{-\alpha_x-1}dx\\ &= p_0^{1-\rho}x_m^{(1-\rho)\varepsilon}\alpha_x x_m^{\alpha_x}\frac{x_m^{(\rho-1)\varepsilon-\alpha_x}}{\alpha_x-(\rho-1)\,\varepsilon}\\ &= p_0^{1-\rho}\frac{\alpha_x}{\alpha_h-1+\varepsilon},\end{aligned}$$

which holds for p_0 as above. Similarly, $\ell(x)$ in (15) satisfies (44) if

$$\begin{aligned} L_w &= L_w\frac{\alpha_h}{\alpha_x}x_m^{1-\rho\varepsilon}\alpha_x x_m^{\alpha_x}\int_{x_m}^{\infty}x^{\rho\varepsilon-1}x^{-\alpha_x-1}dx\\ &= L_w\frac{\alpha_h}{\alpha_x}x_m^{1-\rho\varepsilon}\frac{\alpha_x x_m^{\rho\varepsilon-1}}{\alpha_x+1-\rho\varepsilon},\end{aligned}$$

which holds.

Clearly (12) implies (6) holds. Then since (44) holds, (43) is satisfied if

$$\frac{\alpha_x}{\alpha_h} x_m^{\rho\varepsilon-1} \Psi'(h^*(x)) a_h = x^{\rho\varepsilon-1} f_p(x), \qquad \text{all } x \geq x_m,$$

or using the Pareto densities,

$$\frac{\alpha_x}{\alpha_h} x_m^{\rho\varepsilon-1} \alpha_h \left(a_h x_m\right)^{\alpha_h} \left(a_h x\right)^{-\alpha_h-1} a_h = x^{\rho\varepsilon-1} \alpha_x x_m^{\alpha_x} x^{-\alpha_x-1}, \qquad \text{all } x \geq x_m,$$

which holds for ε in (10). The value for y_F follows from (1). ■

B. The evolution of technology

The distribution function for technology among producers evolves as follows. As noted above, $X_m(t)$ is nondecreasing. Let $\Delta_t > 0$ be a small time increment. For any $t \geq 0$ and any $X \geq X_m(t + \Delta_t)$, the number of producers with technology above X at $t + \Delta_t$ consists of incumbents at t, adjusted for exit, plus successful innovators of both types, selected to include only those with technology greater than $(1 - \mu_x \Delta_t) X$ at date t,

$$\begin{aligned} & \left[1 - F(X, t + \Delta_t)\right] N_p(t + \Delta_t) \\ \approx \ & \left\{1 - F[(1 - \mu_x \Delta_t) X, t]\right\} \left[(1 - \delta_x \Delta_t) N_p(t) + \lambda_{xi} \Delta_t N_i(t) + \Lambda_{xe}(t) \Delta_t N_e(t)\right]. \end{aligned}$$

Taking a first-order approximation gives

$$\begin{aligned} & \left[1 - F(X, t)\right] \left[N_p(t) + N_p'(t) \Delta_t\right] - F_t(X, t) \Delta_t N_p(t) \\ & \qquad \approx \left[1 - F(X, t)\right] \left[(1 - \delta_x \Delta_t) N_p(t) + \lambda_{xi} \Delta_t N_i(t) + \Lambda_{xe}(t) \Delta_t N_e(t)\right] \\ & \qquad \quad + f(X, t) \mu_x \Delta_t X N_p(t). \end{aligned}$$

Collecting terms and dividing by $\Delta_t N_p(t)$ gives the equation in the text.

APPENDIX B: BGPS WITH PARETO DISTRIBUTIONS

A. Firms: process and product innovation

If Π and V^f have the forms required for a BGP, then factoring out $Q_0 e^{g_\pi t}$, the Bellman equation for a producing firm is

$$\begin{aligned}(r+\delta_x)\, v_{fp}(X/Q(t)) &= \pi(X/Q(t)) + \mu_x \frac{X}{Q(t)} v'_{fp}(X/Q(t)) \\ &\quad + g_\pi v_{fp}(X/Q(t)) - v'_{fp}(X/Q(t)) \frac{X}{Q(t)} \frac{\dot{Q}(t)}{Q(t)},\end{aligned}$$

or

$$(r+\delta_x - g_\pi)\, v_{fp}(x) = \pi(x) + (\mu_x - g)\, x v'_f(x),$$

where $x = X/Q$ and $\dot{Q}/Q = g$. For π as in (15), the normalized Bellman equation is as claimed.

Define

$$\begin{aligned} B_x &\equiv \left[(r+\delta_x - g_\pi) + (g-\mu_x)(1-\zeta)\right]^{-1}, \\ R_x &\equiv -\frac{r+\delta_x - g_\pi}{g-\mu_x}, \end{aligned} \tag{45}$$

where $B_x > 0$ and where the $R_x < 0$ is the characteristic root of the ODE. It is straightforward to verify that a particular solution of the ODE is $v_P(x) = B_x \pi_1 x^{1-\zeta}$. As usual, $v_P(x)$ is the value of the firm if it never invests, operating with its evolving technology until the exit shock arrives.

In addition, there is a homogeneous solution, of the form $v_H(x) = c_x x^{R_x}$, where the coefficient $c_x > 0$ is determined by the value matching condition: the value of a firm at the threshold x_m must equal to the value of an investor, $\lim_{x \to x_m} v_{fp}(x) = v_{f0}$. Hence

$$c_x = x_m^{-R_x}\left(v_{fi} - B_x \pi_1 x_m^{1-\zeta}\right),$$

and $v_{fp}(x)$ is as in (23).

Differentiate (23) to get the smooth pasting condition, which represent the optimal choice of the investment threshold x_m by incumbent producers,

$$0 = v'_{fp}(x_m) = (1-\zeta)\, B_x \pi_1 x_m^{-\zeta} + R_x x_m^{-1} \left(v_{fi} - B_x \pi_1 x_m^{1-\zeta} \right).$$

This condition determines v_{fi}, as in (24).

Substituting for v_{fi} in (23) gives

$$v_{fp}(x) = B_x \pi_1 x_m^{1-\zeta} \left[\left(\frac{x}{x_m} \right)^{1-\zeta} - \frac{1-\zeta}{R_x} \left(\frac{x}{x_m} \right)^{R_x} \right], \qquad x \geq x_m,$$

For any $q > \alpha_x$, integrating w.r.t. the density $f(x) = \alpha_x x_m^{\alpha_x} x^{-\alpha_x - 1}$ gives $\mathrm{E}_{\hat{F}} \left[(x/x_m)^q \right] = \alpha_x / \left(\alpha_x - q \right).$ Hence

$$\begin{aligned}
\mathrm{E}_{\hat{F}} \left[v_{fp}(x) \right] &= B_x \pi_1 x_m^{1-\zeta} \left[\frac{\alpha_x}{\alpha_x + \zeta - 1} - \frac{1-\zeta}{R_x} \frac{\alpha_x}{\alpha_x - R_x} \right], \\
\mathrm{E}_{\hat{F}} \left[\pi(x) \right] &= \pi_1 x_m^{1-\zeta} \frac{\alpha_x}{\alpha_x + \zeta - 1}.
\end{aligned}$$

and

$$\begin{aligned}
\frac{\mathrm{E}_{\hat{F}} \left[v_{fp}(x) \right]}{\mathrm{E}_{\hat{F}} \left[\pi(x) \right]} &= B_x \left[1 - \frac{1-\zeta}{R_x} \frac{\alpha_x + \zeta - 1}{\alpha_x - R_x} \right] \\
&= \left[(r + \delta_x - g_\pi) + (g - \mu_x)(1-\zeta) \right]^{-1} \left[1 - \frac{1-\zeta}{R_x} \frac{\alpha_x + \zeta - 1}{\alpha_x - R_x} \right]
\end{aligned}$$

Similarly, if V_{fi} has the form required for a BGP, then the Bellman equation for the normalized value v_{fi} is as claimed. Substituting for $\mathrm{E}_{\hat{F}} \left[v_{fp}(x) \right]$ and v_{fi}, and factoring out $\pi_1 B_x x_m^{1-\zeta}$, the Bellman equation requires

$$\begin{aligned}
r + \delta_x - g_\pi &= \frac{\lambda_{xi}}{v_{fi}} \left\{ \mathrm{E}_{\hat{F}} \left[v_{fp}(x) \right] - v_{fi} \right\} \\
&= \frac{\lambda_{xi} R_x}{R_x - 1 + \zeta} \left(\frac{\alpha_x}{\alpha_x - 1 + \zeta} - \frac{1-\zeta}{R_x} \frac{\alpha_x}{\alpha_x - R_x} + \frac{1-\zeta}{R_x} - 1 \right) \\
&= \frac{\lambda_{xi} R_x}{R_x - 1 + \zeta} \left(\frac{1-\zeta}{\alpha_x - 1 + \zeta} - \frac{1-\zeta}{\alpha_x - R_x} \right) \\
&= \frac{\lambda_{xi} R_x (1-\zeta)}{R_x - 1 + \zeta} \frac{-R_x + 1 - \zeta}{(\alpha_x - 1 + \zeta)(\alpha_x - R_x)} \\
&= \frac{-R_x (1-\zeta) \lambda_{xi}}{(\alpha_x - 1 + \zeta)(\alpha_x - R_x)},
\end{aligned}$$

as in (25).

B. Workers: investment in human capital on the BGP

The analysis for workers is analogous, except that there is no entry cost. Define

$$\begin{aligned} B_h &\equiv \left[(r+\delta_h-g_w)+(g-\mu_h)(1-\varepsilon)\right]^{-1}, \\ R_h &\equiv -\frac{r+\delta_h-g_w}{g-\mu_h}<0. \end{aligned} \tag{46}$$

Using (30) in (29) gives

$$v_w(h)=B_h w_1 h_m^{1-\varepsilon}\left[\left(\frac{h}{h_m}\right)^{1-\varepsilon}-\frac{1-\varepsilon}{R_h}\left(\frac{h}{h_m}\right)^{R_h}\right], \qquad h\geq h_m,$$

and the Pareto form for $\hat{\Psi}$ implies

$$\mathrm{E}_{\hat{\Psi}}\left[v_w(h)\right]=B_h w_1 h_m^{1-\varepsilon}\left[\frac{\alpha_h}{\alpha_h-1+\varepsilon}-\frac{1-\varepsilon}{R_h}\frac{\alpha_h}{\alpha_h-R_h}\right].$$

C. Flows of firms, the DF for technology

On a BGP $X_m(t)$ grows at the rate g; $N_p(t), N_i(t)$, and $N_e(t)$ grow at the rate n; and there is strictly positive process innovation, so (16) holds. Hence the law of motion for N_p requires

$$\begin{aligned} nN_p &= \lambda_{xi}N_i+\lambda_{xe}N_e-\delta_x N_p-(g-\mu_x)\frac{X_m(t)}{Q(t)}f(x_m)N_p \\ &= \lambda_{xi}N_i+\lambda_{xe}N_e-\left[\delta_x+\alpha_x(g-\mu_x)\right]N_p, \end{aligned}$$

where the second line uses the fact that f is a Pareto density with parameters (α_x, x_m). Hence

$$\left[n+\delta_x+\alpha_x(g-\mu_x)\right]N_p=\lambda_{xi}N_i+\lambda_{xe}N_e.$$

The laws of motion for N_i and N_e require

$$\begin{aligned} (n+\delta_x+\lambda_{xi})N_i &= \alpha_x(g-\mu_x)N_p, \\ (n+\delta_x+\lambda_{xe})N_e &= E. \end{aligned}$$

Sum the three laws of motion to get (33), which determines the entry rate E. The population shares for firms are

$$\begin{aligned}
\frac{N_p}{N} &= \frac{n+\delta_x+\lambda_{xi}}{(n+\delta_x+\lambda_{xi})+\alpha_x(g-\mu_x)}\frac{\lambda_{xe}}{n+\delta_x+\lambda_{xe}}, \\
\frac{N_i}{N} &= \frac{\alpha_x(g-\mu_x)}{(n+\delta_x+\lambda_{xi})+\alpha_x(g-\mu_x)}\frac{\lambda_{xe}}{n+\delta_x+\lambda_{xe}}, \\
\frac{N_e}{N} &= \frac{n+\delta_x}{n+\delta_x+\lambda_{xe}},
\end{aligned}$$

and the ratios N_i/N_p and N_e/N_p satisfy (34).

As a check, note that (34) implies

$$\lambda_{xi}\frac{N_i}{N_p}+\lambda_{xe}\frac{N_e}{N_p}-n-\delta_x=\alpha_x(g-\mu_x).$$

Using this expression in the law of motion for F, we get

$$-F_t(X,t)=f(X,t)\mu_x X+[1-F(X,t)]\,\alpha_x(g-\mu_x), \qquad \text{all } X\geq X_m(t), \quad \text{all } t.$$

If F has the form required for a BGP, then

$$\begin{aligned}
f(X,t) &= f(X/Q(t))\;/\;Q(t), \\
-F_t(X,t) &= f(X/Q(t))\,gX/Q(t), \qquad \text{all } X\geq X_m(t), \quad \text{all } t.
\end{aligned}$$

so the required condition is

$$(g-\mu_x)\,xf(x)=(g-\mu_x)\,\alpha_x[1-F(x)], \qquad \text{all } x\geq x_m,$$

which holds since F is a Pareto distribution with parameters (α_x, x_m).

D. Flows of individuals, the DF for skill

The argument for workers parallels the one for firms, except that the entry rate is exogenous. On a BGP, $H_m(t)$ grows at the rate g; $L_w(t), L_i(t), L_e(t)$ all grow at the rate υ; and there is positive investment by individuals, so (17) holds. Hence the law of motion for L_w requires

$$\upsilon L_w = \lambda_{hi} L_i + \lambda_{he} L_e - \left[\delta_h + \alpha_h \left(g - \mu_h\right)\right] L_w.$$

Then

$$\left[\upsilon + \delta_h + \alpha_h \left(g - \mu_h\right)\right] L_w = \lambda_{hi} L_i + \lambda_{he} L_e,$$

the shares of workers, retoolers and entrants in the population are

$$\begin{aligned}
\frac{L_w}{L} &= \frac{\upsilon + \delta_h + \lambda_{hi}}{\left(\upsilon + \delta_h + \lambda_{hi}\right) + \alpha_h \left(g - \mu_h\right)} \frac{\lambda_{he}}{\upsilon + \delta_h + \lambda_{he}}, \\
\frac{L_i}{L} &= \frac{\alpha_h \left(g - \mu_h\right)}{\left(\upsilon + \delta_h + \lambda_{hi}\right) + \alpha_h \left(g - \mu_h\right)} \frac{\lambda_{he}}{\upsilon + \delta_h + \lambda_{he}}, \\
\frac{L_e}{L} &= \frac{\upsilon + \delta_h}{\upsilon + \delta_h + \lambda_{he}},
\end{aligned}$$

and the ratios L_i/L_w and L_e/L_w satisfy (35).

Since Ψ is a Pareto distribution with parameters $\left(\alpha_h, h_m\right)$, the law of motion for Ψ satisfies the required condition.

APPENDIX C: WAGE AND FIRM SIZE DYNAMICS

As a function of age, the shares of firms of various types satisfy

$$\begin{pmatrix} \sigma_p'(a) \\ \sigma_i'(a) \\ \sigma_e'(a) \end{pmatrix} = \begin{pmatrix} -\alpha_x \left(g - \mu_x\right) & \lambda_{xi} & \lambda_{xe} \\ \alpha_x \left(g - \mu_x\right) & -\lambda_{xi} & 0 \\ 0 & 0 & -\lambda_{xe} \end{pmatrix} \begin{pmatrix} \sigma_p(a) \\ \sigma_i(a) \\ \sigma_e(a) \end{pmatrix}.$$

It is straightforward to solve this linear system, and find that

$$\begin{aligned}
\sigma_p(a) &= c_1 \lambda_{xi} + e^{-b_x a} \left(c_1 + c_3\right) \alpha_x \left(g - \mu_x\right) + e^{-\lambda_{xe} a} c_3 \left(\lambda_{xi} - \lambda_{xe}\right), \\
\sigma_i(a) &= c_1 \alpha_x \left(g - \mu_x\right) - e^{-b_x a} \left(c_1 + c_3\right) \alpha_x \left(g - \mu_x\right) + e^{-\lambda_{xe} a} c_3 \alpha_x \left(g - \mu_x\right), \\
\sigma_e(a) &= 0 + 0 + e^{-\lambda_{xe} a},
\end{aligned}$$

where $b_x \equiv \alpha_x \left(g - \mu_x\right) + \lambda_{xi}$, $c_1 = 1/b_x$, and $c_3 = -1/\left(b_x - \lambda_{xe}\right)$.

Figure 1a: comparative static (a)

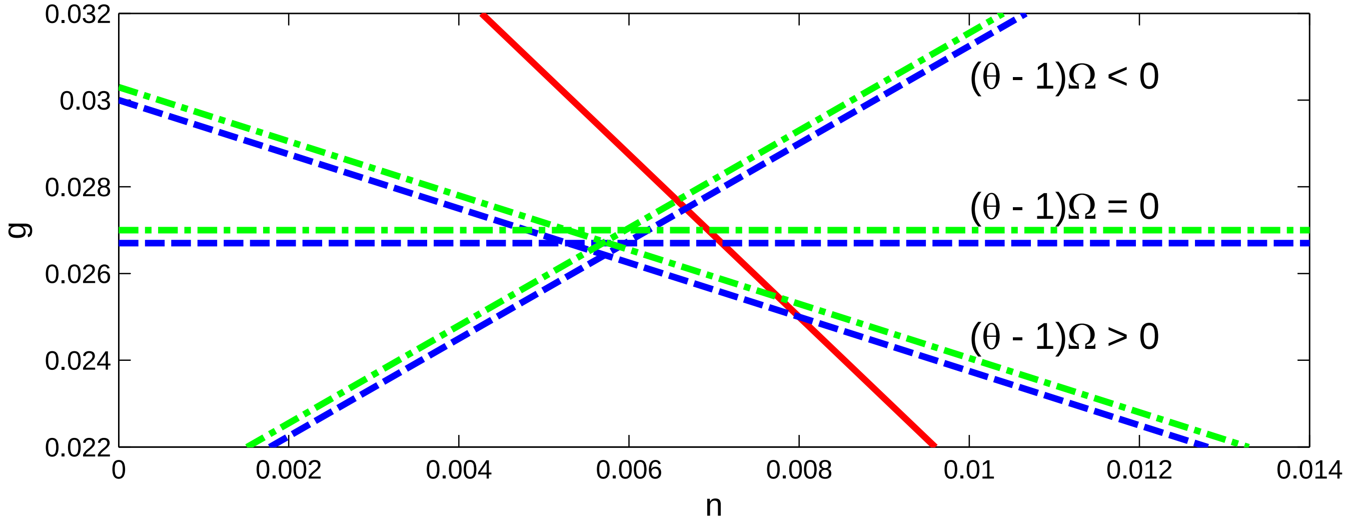

Figure 1b: comparative static (b)

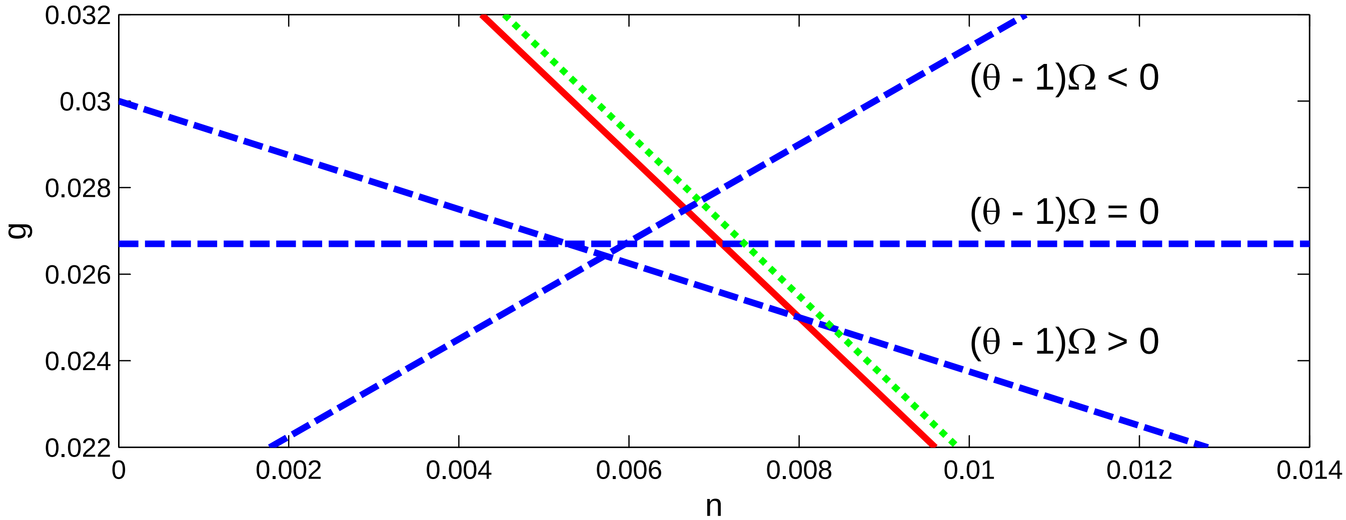

Figure 1c: comparative static (c)