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Catching up and falling behind

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Abstract This paper studies the interaction between technology, a public input that flows in from abroad, and human capital, a private input that is accumulated domestically, as twin engines of growth in a developing economy. The model displays two types of long run behavior, depending on policies and initial conditions. One is sustained growth, where the economy keeps pace with the technology frontier. The other is stagnation, where the economy converges to a minimal technology level that is independent of the world frontier. In a calibrated version of the model, transition paths after a policy change can display rapid growth, as in modern growth 'miracles.' In these economies policies that promote technology inflows are much more effective than subsidies to human capital accumulation in accelerating growth. A policy reversal produces a 'lost decade,' a period of slow growth that permanently reduces the level of income and consumption.

Keywords Growth · Technology · Human capital · Miracle · Lost decade

JEL Classification O40 · O38

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This paper develops a model of growth that can accommodate the enormous differences in observed outcomes across countries and over time: periods of rapid growth as less developed countries catch up to the income levels of those at the frontier, long periods of sustained growth in developed countries, and substantial periods of decline in countries that at one time seemed to be catching up. The sources of growth are technology, which flows in from

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abroad, and human capital, which is accumulated domestically. The key feature of the model is the interaction between these two forces.

Human capital is modeled here as a private input into production, accumulated using the agent's own time (current human capital) and the local technology as inputs. Thus, improvements in the local technology affect human capital accumulation directly, by increasing the productivity of time spent in that activity. Improvements in the local technology also raise the local wage, increasing both the benefits and costs of time spent accumulating human capital. If the former outweigh the latter, human capital accumulation is stimulated through this channel as well.

In the reverse direction, human capital affects the inflow rate of technology, a public input into production. This channel represents many specific mechanisms. For example, better educated entrepreneurs and managers are better able to identify new products and processes that are suitable for the local market. In addition, a better educated workforce makes a wider range of new products and processes viable for local production, an important consideration for both domestic entrepreneurs interested in producing locally and foreign multinationals seeking attractive destinations for direct investment. Because local human capital has this positive external effect, public subsidies to its accumulation are warranted, and here they can affect the qualitative behavior of the economy in the long run.

The framework here shares several features with the one in Parente and Prescott (1994), including a world technology frontier that grows at a constant rate and "barriers" that impede the inflow of new technologies into particular countries. The local technology is modeled here as a pure public good, with the rate of technology inflow governed by three factors: (i) the domestic technology gap, relative to a world frontier, (ii) the domestic human capital stock, also relative to the world frontier technology, and (iii) the domestic barrier. The growth rate of the local technology is an increasing function of the technology gap, reflecting the fact that a larger pool of untapped ideas offers more opportunities for the adopting country. As noted above, it is also increasing in local human capital, reflecting the role of education in enhancing the ability to absorb new ideas. Finally, the barrier reflects tariffs, internal taxes, capital controls, currency controls, or any other policy measures that retard the inflow of ideas and technologies.

For fixed parameter values, the model displays two types of long run behavior, depending on the policies in place and the initial conditions. If the barrier to technology inflows is low, the subsidy to human capital accumulation is high, and the initial levels for the local technology and local human capital are not too far below the frontier, the economy displays sustained growth in the long run. In this region of policy space, and for suitable initial conditions, higher barriers and lower subsidies do not affect the long-run growth rate, which is equal to the growth rate of the frontier technology. They do, however, imply slower convergence to the economy's balanced growth path (BGP), wider long-run gaps between the local technology and the frontier, and lower levels for capital stocks, output and consumption along the BGP.

Thus, the model predicts that high- and middle-income countries can, over long periods, grow at the same rate as the world frontier. In these countries the relative gap between the local technology and the world frontier is constant in the long run, and not too large.

Alternatively, if the technology barrier is sufficiently high, the subsidy to human capital accumulation is sufficiently low, or some combination, balanced growth is not possible. Instead, the economy stagnates in the long run. An economy with policies in this region converges to a minimal technology level that is independent of the world frontier, and a human capital level that depends on the local technology and the local subsidy. For sufficiently low initial levels of the local technology and human capital, the economy converges to the stagnation steady state, even for policy parameters that permit balanced growth for more favorable initial conditions.

Thus, low-income countries—those with large technology gaps—cannot display modest, sustained growth, as middle- and high-income countries can. They can adopt policies that trigger a transition to a BGP, or they can stagnate, falling ever farther behind the frontier. Moreover, economies that enjoyed technology inflows in the past can experience technological regress if they raise their barriers: local TFP and per capita income decline during the transition to a stagnation steady state.

In the model here two policy parameters affect the economy's performance, the barrier to technology inflows and the subsidy to human capital accumulation. Although both can be used to speed up transitional growth, simulations suggest that stimulating technology inflows is a more powerful tool. The reasons for this are twofold. First, human capital accumulation takes resources away from production, reducing output in the short run. In addition, for reasonable model parameters, human capital accumulation is necessarily slow. Thus, while subsides to its accumulation eventually lead to faster technology inflows and higher productivity, the process is prolonged. Policies that enhance technology inflows increase output immediately. They also increase the returns to human (and physical) capital, thus stimulating further investment and growth in the long run.

The rest of the paper is organized as follows. Section 1 discusses evidence on the importance of technology inflows as a source of growth. It also documents the fact that many countries are not enjoying these inflows, instead falling ever further behind the world frontier. In Sect. 2 the model is described, in Sect. 3 the BGPs and stagnation steady states are characterized, and in Sect. 4 their stability is discussed. Sections 5 and 6 describe the baseline calibration and computational results. Section 7 provides sensitivity analysis for parameters about which there is little direct evidence, and Sect. 8 concludes.

1 Evidence on the sources of growth

Five types of evidence point to the conclusion that differences in technology are critical for explaining differences in income levels over time and across economies. Collectively, they make a strong case that developed economies share a common, growing 'frontier' technology, and that less developed economies grow rapidly by tapping into that world technology.[1]

First, most growth accounting exercises for individual developed countries, starting with those in Solow (1957) and Denison (1974), attribute a large share of the increase in output per worker to an increase in total factor productivity (TFP). Although measured TFP in these exercises, the Solow residual, surely includes the influence of other (omitted) variables, the search for those missing factors has been extensive, covering a multitude of potential explanatory variables, many countries, many time periods, and many years of effort. It is difficult to avoid the conclusion that technical change is a major ingredient. Similarly, Klenow (1998) finds that cross-industry data for US manufacturing supports the view that nonrival ideas—technologies—are important compared with (rival) human capital. Studies like Jorgenson et al. (1987), Bowlus and Robinson (2012), and others that succeed in substantially reducing the Solow residual do so by introducing quality adjustments to physical capital or human capital or both. Arguably these quality adjustments represent technical change, albeit in a different form.

[1] See Prescott (1997), Klenow and Rodríguez-Clare (2005), and Comin and Hobijn (2010) for further evidence supporting this conclusion.

Second, development accounting exercises using cross-country data arrive at a similar conclusion, finding that differences in physical and human capital explain only a modest portion of the differences in income levels across countries. For example, Hall and Jones (1999) find that of the 35-fold difference in GDP per worker between the richest and poorest countries, measured inputs—physical and human capital per worker—account for 4.5-fold, while differences in TFP—the residual—accounts for 7.7-fold. Klenow and Rodríguez-Clare (1997) arrive at a similar conclusion. In addition, Caselli and Feyrer (2007) find that the rate of return on capital is similar across a broad cross-section of countries, suggesting that capital misallocation is not an important source of differences in income.

To be sure, the cross-country studies have a number of limitations. Few countries have data on hours, so output is measured per worker rather than per manhour. Human capital is measured as average years of education in the population, with at best a rough adjustment for educational quality.[2] No adjustment is made for differences in educational attainment in the workforce and in the population as a whole, differences which are probably greater in countries with lower average attainment. Nor is any adjustment made for other aspects of human capital, such as health. And as in growth accounting exercises, the figure for TFP is a residual, so it is surely biased upward. Nevertheless, the role allocated to technology is large enough to absorb a substantial amount of downward revision and survive as a key determinant of cross-country differences.

A third piece of evidence is Baumol's (1986) study of the OECD countries. Although criticized on methodological grounds (see De Long 1988; Baumol and Wolff 1988), the data nevertheless convey an important message: the OECD countries (and a few more) seem to share common technologies. It is hard to explain in any other way the harmony over many decades in both their income levels and growth rates. Moreover, as Prescott (2002, 2004) and Ragan (2013) have shown, much of the persistent differences in income levels can be explained by differences in fiscal policy that affect work incentives.

A fourth piece of evidence for the importance of technology comes from data on "late bloomers." As first noted by Gerschenkron (1962), economies that develop later have an advantage over the early starters exactly because they can adopt technologies, methods of organization, and so on developed by the leaders. Followers can learn from the successes of their predecessors and avoid their mistakes. Parente and Prescott's (1994, 2000) evidence on doubling times makes this point systematically. Figure 1 reproduces their scatter plot, updated to include data through 2008. Each point in this figure represents one of the 65 countries that had reached a per capita GDP of $4,000 by 2008. On the horizontal axis is the year that the country first reached $2,000, and on the vertical axis is the number of years required to first reach $4,000.

As Fig. 1 shows, there is a strong downward trend: countries that arrived at the $2,000 figure later doubled their incomes more quickly. The later developers seem to have enjoyed the advantage of fishing from a richer pool of ideas, ideas provided by advances in world

[2] Even if included, differences in educational quality might have a modest impact, however. Hendricks (2002) reports that many studies find that immigrants' earnings are within 25 % of earnings of native-born workers with the same age, sex, and educational attainment.

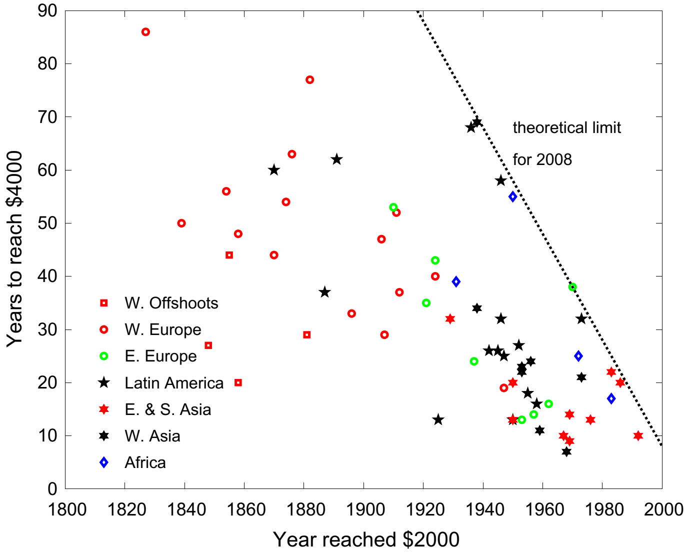

Fig. 1 Doubling times for per capita GDP, 65 countries

technology.[3] Lucas (2009), Ramondo (2009), Hahn (2012) and many others find an important role for international trade in stimulating such inflows.[4]

A fifth and final piece of evidence supporting the importance of technology is the occurrence, infrequently, of "growth miracles." The term is far from precise, and a stringent criterion should be used in classifying countries as such, since growth rates in developing countries show little persistence from one decade to the next. Indeed, mean reversion in income levels following a financial crisis or similar event implies that an especially bad decade in terms of growth rates is likely to be followed by a good one. But recovery from a disaster is not a miracle.

Nevertheless, over the period 1950–2006 twelve countries (i) enjoyed at least one 20-year episode where average per capita GDP growth exceeded 5 %, and (ii) in 2006 had GDP per capita that was at least 45 % of the US. This group has five members in Europe (Germany, Italy, Greece, Portugal, and Spain), five in east Asia (Japan, Taiwan, Hong Kong, Singapore, and Korea), and two others (Israel and Puerto Rico). The jury is still out on several others candidates: China, Thailand, Malaysia, and Botswana have met criterion (i) but not (yet) accomplished (ii).[5]

[3] The data used here are from Maddison (2010), with oil producers and countries with population under one million in 1960 omitted. Figure 1 has two biases, which should be noted. First, there are eight countries for which the first observation is in 1950, and it exceeds $2,000. Some of these points should be shifted to the left, which would strengthen the downward trend. Second, among countries that reached the $2,000 figure by the year 2000, the slow growers have not yet reached $4,000. The dotted line indicates a region that by construction is empty. Ignoring the pool of countries that in the future will occupy this space biases the impression in favor of the 'backwardness' hypothesis.

[4] See Acemoglu and Zilibotti (2001) for a contrarian view.

[5] It is sobering to see how many countries enjoyed 20-year miracles, yet gave up all their (relative) gains or even lost ground over the longer period. Among countries in this groups are Bulgaria, Yugoslavia, Jamaica, North Korea, Iran, Gabon, Libya, and Swaziland.

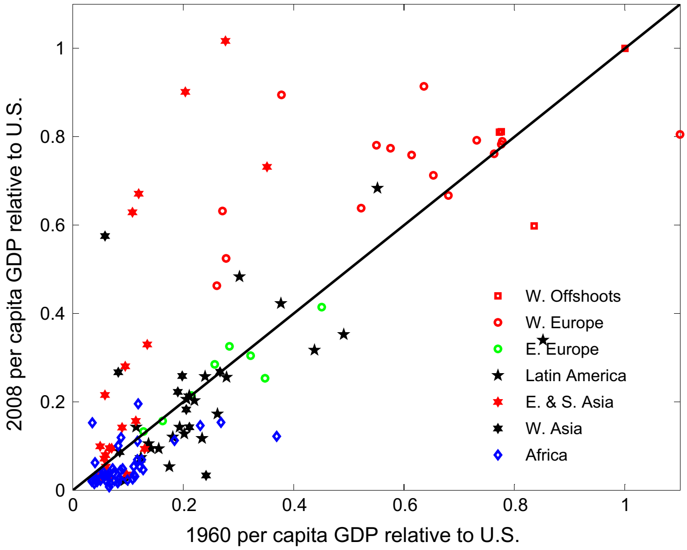

Fig. 2 Catching up and falling behind, 1960–2008

Although each of these five sources of evidence has individual weaknesses, taken together they make a strong case for the importance of international technology spillovers in keeping income levels loosely tied together in the developed countries, and occasionally allowing a less developed country to enjoy a growth spurt during which it catches up to the more developed group.

Not all countries succeed in tapping into the global technology pool, however. Figure 2 shows the world pattern of catching up and falling behind, with the US taken as the benchmark for growth. It plots per capita GDP relative to the US in 2008, against per capita GDP relative to the US in 1960, for 127 countries. Countries that are above the 45° line have gained ground over that 48-year period, and those below it have lost ground. It is striking how few have gained. The geographic pattern of gains and losses is striking as well. The countries that are catching up are almost exclusively European (plus Israel) and East Asian. With only a few exceptions, countries in Latin America, Africa, and South Asia have fallen behind.

Figure 3 shows 70 of the poorest countries, those in Asia and Africa, in more detail. The six Asian miracles (Israel, Japan, Taiwan, Hong Kong, Singapore, and Korea) are omitted, since they significantly alter the scale. Here the plot shows per capita GDP in 2008 against per capita GDP in 1960, both in levels. The rays from the origin correspond to various average growth rates. Keeping pace with the US over this period requires a growth rate of 2 %. The number of countries that have gained ground relative to the US (points above the 2 % growth line) is modest, while the number that have fallen behind (points below the line) is much larger. A shocking number, mostly in Africa, have suffered negative growth over the whole period.

The model developed below focuses on technology inflows as the only source of sustained growth. Other factors are neglected, although some are clearly important.

For example, it is well documented that in most developing economies, TFP in agriculture is substantially lower than it is in the non-agricultural sector.[6] Thus, an important component

[6] See Caselli (2005) for recent evidence that TFP differences across countries are much greater in agriculture than they are in the non-agricultural sector.

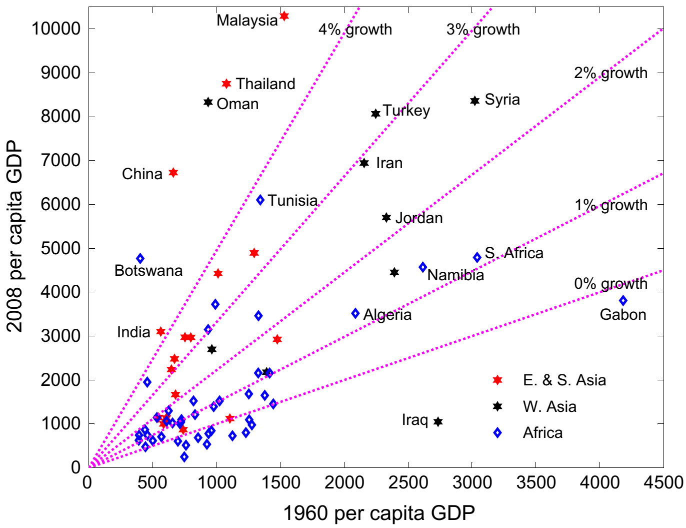

Fig. 3 Asia and Africa, excluding six successes

of growth in almost every fast-growing economy has been the shift of labor out of agriculture and into other occupations. The effect of this shift on aggregate TFP, which is significant, cannot be captured in a one-sector model.

More recently, detailed data for manufacturing has allowed similar estimates for the gains from re-allocation across firms within that single sector. Such misallocation can result from financial market frictions, from frictions that impede labor mobility, or from taxes (or other policies) that distort factor prices. For China, Hsieh and Klenow (2009) find that improvements in allocative efficiency contributed about 1/3 of the 6.2 % TFP growth in manufacturing over the period 1993–2004.

Although this gain in Chinese manufacturing is substantial, it is a modest part of overall TFP growth in China—in all sectors—over that period. In addition, evidence from other countries gives reallocation a smaller role. Indeed, Hsieh and Klenow find that in India allocative efficiency declined over the same period, although per capita income grew. Similarly, Bartelsman et al. (2013) find that in the two countries—Slovenia and Hungary—for which they have time series, most of the (substantial) TFP gains those countries enjoyed during the 1990s came from other sources: improvements in allocative efficiency played a minor role. And most importantly, TFP gains from resource reallocation are one-time gains, not a recipe for sustained growth.[7]

The model here is silent about the source of advances in the technology frontier, which are taken as exogenous. Thus, it is complementary to the large body of work looking at

[7] In addition, it is not clear what the standard for allocative efficiency should be in a fast-growing economy. Restuccia and Rogerson (2008) develop a model with entry, exit, and fixed costs that produces a non-degenerate distribution of productivity across firms, even in steady state. Their model has the property that the stationary distribution across firms is sensitive to the fixed cost of staying in business and the distribution of productivity draws for potential entrants. There is no direct evidence for either of these important components, although they can be calibrated to any observed distribution. Thus, it is not clear if differences across countries reflect distortions that affect the allocation of factors, or if they represent differences in fundamentals, especially in the 'pool' of technologies that new entrants are drawing from. In particular, the distribution of productivities for new entrants might be quite different in a young, fast-growing economy like China and a mature, slow-growing economy like the US.

investments in R&D, learning-by-doing, and other factors that affect the pace of innovation at the frontier. Nor does it say anything about the societal factors that lead countries to develop institutions or adopt policies that stimulate growth, by stimulating technology inflows, encouraging factor accumulation, and so on. Thus, it is also complementary to the work of Acemoglu et al. (2001, 2002, 2005) and others, that looks at the country characteristics associated with economic success, without specifying the more proximate mechanism.

2 The model

The representative household has preferences over intertemporal consumption streams. It allocates its time between work and human capital accumulation, and its income between consumption and savings, taking as given the paths for wages, interest rates, and the local technology, as well as the subsidy to human capital accumulation. The representative firm operates a constant-returns-to-scale (CRS) technology, taking as given the path for local TFP. It hires capital and labor, paying them their marginal products. The government makes one-time choices about two policy parameters, the barrier to technology inflows and the subsidy to human capital accumulation. It finances the subsidy with a lump sum tax, maintaining budget balance at all times. Technology inflows depend on the average human capital level, as well as the barrier. In this section we will first describe the technology inflows in more detail, and then turn to the household's and firm's decisions.

2.1 The local technology

The model of technology is a variant of the diffusion model first put forward in Nelson and Phelps (1966) and subsequently developed elsewhere in many specific forms.[8] There is a frontier (world) technology $W(t)$ that grows at a constant, exogenously given rate,

$$\dot{W}(t) = gW(t), \tag{1}$$

where $g > 0$. In addition, each country i has a local technology $A_i(t)$. Growth in $A_i(t)$ is described by

$$\begin{aligned} \frac{\dot{A}_i}{A_i} &= 0, \qquad \text{if } A_i = A_i^{st} \text{ and } \frac{\psi_0}{B_i}\frac{\overline{H}_i}{W}\left(1 - \frac{A_i^{st}}{W}\right) < \delta_A, \\ \frac{\dot{A}_i}{A_i} &= \frac{\psi_0}{B_i}\frac{\overline{H}_i}{W}\left(1 - \frac{A_i}{W}\right) - \delta_A, \qquad \text{otherwise,} \end{aligned} \tag{2}$$

where A_i^{st} is a lower bound on the local technology, B_i is a policy parameter, $\overline{H}_i$ is average human capital, $\delta_A > 0$ is the depreciation rate for technology, and $\psi_0 > 0$ is a constant.

The technology floor A_i^{st} allows the economy to have a 'stagnation' steady state with a constant technology. Above that floor, technology growth is proportional to the ratio $\overline{H}_i/W$ of local human capital to the frontier technology, and to the gap $(1 - A_i/W)$ between the local technology and the frontier.[9] The former measures the capacity of the economy to absorb technologies near the frontier, while the latter measures the pool of technologies that have not yet been adopted. A higher value for the frontier technology W thus has two effects.

[8] See Benhabib and Spiegel (2005) for an excellent discussion of the long-run dynamics of various versions.

[9] See Benhabib et al. (2014) for a model where imitation of this type is costly, but for technological laggards is less costly than innovation.

It widens the technology gap, which tends to speed up growth, but also reduces the absorption capacity, which tends to retard growth. The first effect dominates if the ratio A_i/W is high and the second if H_i/W is low, resulting in a logistic form.[10]

As in Parente and Prescott (1994), the barrier $B_i \geq 1$ can be interpreted as an amalgam of policies that impede access to or adoption of new ideas, or reduce the profitability of adoption. For example, it might represent impediments to international trade that reduce contact with new technologies, taxes on capital equipment that is needed to implement new technologies, poor infrastructure for electric power or transportation, or civil conflict that impedes the cross-border flow of people and ideas. Notice that because of depreciation, technological regress is possible. Thus, an increase in the barrier can lead to the abandonment of once-used technologies.

2.2 Households and firms

Next consider the decisions of households and firms. There is a continuum of identical households, and population is constant. Households accumulate physical capital, which they rent to firms, and in addition each household is endowed with one unit of time (a flow), which it allocates between human capital accumulation and goods production. Investment in human capital uses the household's own time and human capital H_i as inputs, as well as the local technology. Economy-wide average human capital, $\overline{H}_i$, may also play a role. In particular,

$$\dot{H}_i(t) = \phi_0 \left[v_i(t) H_i(t)\right]^{\eta} A_i(t)^{\zeta} \overline{H}_i(t)^{1-\eta-\zeta} - \delta_H H_i(t), \tag{3}$$

where v_i is time allocated to human capital accumulation, $\delta_H > 0$ is the depreciation rate for human capital, and $\eta, \zeta > 0, \quad \eta + \zeta \leq 1$. This technology has constant returns to scale jointly in the stocks $\left(A_i, H_i, \overline{H}_i\right)$, which permits sustained growth. The restriction $\zeta > 0$ rules out sustained growth in the absence of technology diffusion, and the restriction $\eta < 1$ implies an Inada condition as $v_i \to 0$, so the time allocated to human capital accumulation is always strictly positive. Average human capital $\overline{H}_i$ plays a role if $\eta + \zeta < 1$, but this channel may be absent.

The technology for producing effective labor has a similar form. Specifically, the household's effective labor supply is

$$L_i(t) = \left[1 - v_i(t)\right] H_i(t)^{\omega} A_i(t)^{\beta} \overline{H}_i(t)^{1-\beta-\omega}, \tag{4}$$

where $(1 - v_i)$ is time allocated to goods production, and $\omega, \beta > 0, \ \omega + \beta \leq 1$. Here the time input enters linearly, but an Inada condition on utility insures that time allocated to production is always strictly positive. Here, too, average human capital $\overline{H}_i$ may play a role, as in Lucas (1988).

The technology for goods production is Cobb–Douglas, with physical capital K_i and effective labor L_i as inputs,

$$Y_i = K_i^{\alpha} L_i^{1-\alpha},$$

[10] Benhabib and Spiegel (2005, Table 2) find that cross-country evidence on the rate of TFP growth supports the logistic form: countries with very low TFP also have slower TFP growth. Their evidence also seems to support the inclusion of a depreciation term.

so factor returns are

$$R_i(t) = \alpha \left(\frac{K_i}{L_i}\right)^{\alpha-1},$$
$$\hat{w}_i(t) \equiv (1-\alpha)\left(\frac{K_i}{L_i}\right)^{\alpha}, \tag{5}$$

where $\hat{w}$ is the return to a unit of effective labor. Hence the wage for a worker with human capital H_i is

$$w_i(H_i, t) = \hat{w}_i H_i^{\omega} A_i^{\beta} \overline{H}_i^{1-\beta-\omega}. \tag{6}$$

Average human capital has a positive external effect through its impact on technology inflows and, possibly, on human capital accumulation and goods production as well. Hence there is a case for public subsidies to its accumulation. To limit the policy to one parameter, the subsidy is assumed to have the following form. Time spent investing in human capital is subsidized at the rate $\sigma_i w_i(\overline{H}_i, t)$, where $\sigma_i \in [0, 1)$ is the policy variable and $\overline{H}_i$ is average human capital in the economy. Thus, the subsidy is a fraction of the current average wage, a form that permits balanced growth. The subsidy is financed with a lump sum tax τ_i, and the government's budget is balanced at all dates, so

$$\tau_i(t) = \overline{v}_i(t)\sigma_i w_i(\overline{H}_i, t), \qquad t \geq 0, \tag{7}$$

where $\overline{v}_i(t)$ is the average time allocated to human capital accumulation.

Thus, the household's budget constraint is

$$\dot{K}_i(t) = (1-v_i)\, w_i(H_i, t) + v_i\sigma_i w_i(\overline{H}_i, t) + (R_i - \delta_K)\, K_i - C_i - \tau_i, \tag{8}$$

where C_i is consumption, K_i is the capital stock, and $\delta_K > 0$ is the depreciation rate for physical capital. Households have constant elasticity preferences, with parameters $\rho, \theta > 0$. Hence the household's problem is

$$\max_{\{v_i(t), C_i(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{C_i^{1-\theta}(t)}{1-\theta} dt \qquad \text{s.t. (3) and (8)}, \tag{9}$$

given σ_i, (H_{i0}, K_{i0}), and $\left\{A_i(t), \overline{H}_i(t), R_i(t), w_i(\cdot, t), \tau_i(t), \ \ t \geq 0\right\}$.

2.3 Competitive equilibrium

A competitive equilibrium requires utility maximization by households, profit maximization by firms, and budget balance for the government. In addition, the law of motion for the local technology must hold.

Definition Given the policy parameters B_i and σ_i and initial values for the state variables W_0, A_{i0}, H_{i0}, and K_{i0}, a *competitive equilibrium* consists of $\{A_i(t), H_i(t), \overline{H}_i(t),$ $K_i(t), v_i(t), C_i(t), R_i(t), w_i(\cdot, t), \tau_i(t), \ \ t \geq 0\}$ with the property that:

(i) $\{v_i, C_i, H_i, K_i\}$ solves (9), given σ_i, (H_{i0}, K_{i0}), and $\left\{A_i, \overline{H}_i, R_i, w_i, \tau_i\right\}$;
(ii) $\{R_i, w_i\}$ satisfies (5) and (6), and $\{\tau_i\}$ satisfies (7);
(iii) $\{A_i\}$ satisfies (2) and $\left\{\overline{H}_i(t) = H_i(t), \ \ t \geq 0\right\}$.

The system of equations characterizing the equilibrium is in the Appendix.

Two types of behavior are possible in the long run. If the barrier B_i is low enough and/or the subsidy σ_i is high enough, balanced growth is possible. Specifically, if the policy parameters

(B_i, σ_i) lie inside a certain set, the economy has two balanced growth paths (BGPs), on which the time allocation v_i^{bg} is constant and the variables (A_i, H_i, K_i, C_i) all grow at the rate g. On BGPs, the ratios A_i/W, H_i/W, and so on depend on B_i and σ_i.

Balanced growth is not the only possibility, however. For any policy parameters (B_i, σ_i) the economy has a unique stagnation steady state (SS) with no growth. In a SS, the local technology is A_i^{st}, and the variables $\left(v_i^{st}, H_i^{st}, K_i^{st}, C_i^{st}\right)$ are constant. Their levels depend on the subsidy σ_i, but not on the barrier B_i. The next section describes the BGPs and stagnation SS in more detail. Before proceeding, however, it is useful to see how growth accounting and development accounting work in this setup.

2.4 Growth and development accounting

Consider a world with many economies, $j = 1, 2, \ldots, J$. In each economy j, output per capita at any date t is

$$Y_j(t) = K_j(t)^{\alpha} \left\{\left[1 - v_j(t)\right] H_j(t)^{1-\beta} A_j(t)^{\beta}\right\}^{1-\alpha}.$$

Assume that at each date, any individual is engaged in only one activity, and hours per worker are the same over time and across countries. Then all differences in v—in the allocation of time between production and human capital accumulation—take the form of differences in labor force participation. Let

$$\hat{Y}_j(t) \equiv \frac{Y_j(t)}{1 - v_j(t)} \qquad \text{and} \qquad \hat{K}_j(t) \equiv \frac{K_j(t)}{1 - v_j(t)}$$

denote output and capital per worker, and note that human capital H_j is already so measured. Then output per worker can be written three ways,

$$\begin{aligned} \hat{Y}_j(t) &= \hat{K}_j(t)^{\alpha} \left[H_j(t)^{1-\beta} A_j(t)^{\beta}\right]^{1-\alpha}, \\ \hat{Y}_j(t) &= \left(\frac{\hat{K}_j(t)}{\hat{Y}_j(t)}\right)^{\alpha/(1-\alpha)} H_j(t)^{1-\beta} A_j(t)^{\beta}, \\ \hat{Y}_j(t) &= \left(\frac{\hat{K}_j(t)}{\hat{Y}_j(t)}\right)^{\alpha/\beta(1-\alpha)} \left(\frac{H_j(t)}{\hat{Y}_j(t)}\right)^{(1-\beta)/\beta} A_j(t). \end{aligned} \tag{10}$$

Suppose that α and β are known, and that H_j is observable, as well as $\hat{Y}_j$ and $\hat{K}_j$. The first equation in (10) is the standard basis for a growth accounting exercise, as in Solow (1957); the second is the version used for the development accounting exercises in Hall and Jones (1999), Hendricks (2002), and elsewhere; and the third is a variation suitable for the model here. In each case the technology level A_j is treated as a residual.

First consider a single economy. A growth accounting exercise based on the first line in (10) in general attributes growth in output per worker to growth in all three inputs, $\hat{K}_j$, H_j and A_j, and along a BGP—where all four variables grow at a common, constant rate—the shares are α, $(1-\beta)(1-\alpha)$, and $\beta(1-\alpha)$.

An accounting exercise based on the second line attributes some growth to physical capital only if the ratio $\hat{K}_j/\hat{Y}_j$ is growing. The rationale for using the ratio is that growth in H_j or A_j induces growth in $\hat{K}_j$, by raising its return. Here the accounting exercise attributes to growth in capital only increases in excess of those prompted by growth in effective labor. Along a BGP $\hat{K}_j/\hat{Y}_j$ is constant, and the exercise attributes all growth to H_j and A_j, with shares $(1-\beta)$ and β.

The third line applies the same logic to human capital, since growth in A_j induces growth in H_j as well as K_j. Since $H_j/\hat{Y}_j$ is also constant along a BGP, here the accounting exercise attributes all growth on a BGP to the residual A_j.

The same logic applies in a development accounting exercise involving many countries. Differences across countries in capital taxes, public support to education, and other policies lead to differences in the ratios $\hat{K}_j/\hat{Y}_j$ and $H_j/\hat{Y}_j$, so a development accounting exercise using any of the three versions attributes some differences in labor productivity to differences in physical and human capital. But the second line attributes to physical capital—and the third to both types of capital—only differences in excess of those induced by changes in the supplies of the complementary factor(s), H_j and A_j in the second line and A_j in the third.

3 Balanced growth paths and steady states

In this section the BGPs for growing economies and the SS for an economy that stagnates are characterized. It is convenient to analyze BGPs in terms of variables that are normalized by the world technology, A_i/W, H_i/W, and so on, while it is convenient to analyze the SS in terms of the levels A_i, H_i, and so on. It will be important to keep this in mind in the next section, where transitions are discussed.

3.1 BGPs for economies that grow

For convenience drop the country subscript. It is easy to show that on a BGP the variables A, H, K, L, and C grow at the same rate g as the frontier technology, the time allocation $v > 0$ is constant, the factor returns R and $\hat{w}$ are constant, the average wage per manhour $w(\overline{H})$ grows at the rate g, and the costate variables Λ_H and Λ_K for the household's problem grow at the rate $-\theta g$.

Thus, to study growing economies it is convenient to define the normalized variables

$$\begin{aligned} a(t) &\equiv \frac{A(t)}{W(t)}, \qquad h(t) \equiv \frac{H(t)}{W(t)}, \qquad \lambda_h(t) \equiv \frac{\Lambda_H(t)}{W^{-\theta}(t)} \\ c(t) &\equiv \frac{C(t)}{W(t)}, \qquad k(t) \equiv \frac{K(t)}{W(t)}, \qquad \lambda_k(t) \equiv \frac{\Lambda_K(t)}{W^{-\theta}(t)}, \text{ all } t. \end{aligned} \tag{11}$$

Using the fact that in equilibrium $\overline{h} = h$, the equilibrium conditions can then be written as

$$\begin{aligned} \lambda_h \phi_0 \eta v^{\eta-1} &= (1-\sigma)\,\lambda_k \hat{w} \left(\frac{a}{h}\right)^{\beta-\zeta}, \\ c^{-\theta} &= \lambda_k, \\ \frac{\dot{\lambda}_h}{\lambda_h} &= \rho + \theta g + \delta_H - \phi_0 \eta v^{\eta} \left(\frac{a}{h}\right)^{\zeta} \left(1 + \frac{\omega}{1-\sigma}\frac{1-v}{v}\right), \\ \frac{\dot{\lambda}_k}{\lambda_k} &= \rho + \theta g + \delta_K - R, \\ \frac{\dot{h}}{h} &= \phi_0 v^{\eta} \left(\frac{a}{h}\right)^{\zeta} - \delta_H - g, \\ \frac{\dot{k}}{k} &= \kappa^{\alpha-1} - \frac{c}{k} - \delta_K - g, \\ \frac{\dot{a}}{a} &= \frac{\psi_0}{B} h\,(1-a) - \delta_A - g, \end{aligned} \tag{12}$$

where

$$\kappa \equiv K/L = k/(1-v)\, a^{\beta} h^{1-\beta},$$
$$R = \alpha \kappa^{\alpha-1}, \qquad \hat{w} = (1-\alpha)\,\kappa^{\alpha}. \tag{13}$$

The transversality conditions hold if and only if $\rho > (1-\theta)\,g$, which insures that the discounted value of lifetime utility is finite.

Let a^{bg}, h^{bg}, and so on denote the constant values for the normalized variables along the BGP. As usual, the interest rate is

$$r^{bg} \equiv R^{bg} - \delta_K = \rho + \theta g,$$

so the transversality condition implies that the interest rate exceeds the growth rate, $r^{bg} > g$. Also as usual, the input ratio κ^{bg} is determined by the rate of return condition

$$\alpha \left(\kappa^{bg}\right)^{\alpha-1} = R^{bg} = r^{bg} + \delta_K.$$

The return to effective labor $\hat{w}^{bg}$ then depends on κ^{bg}. Thus, all economies on BGPs have the same growth rate g and interest rate r^{bg}, which do not depend on the policy parameters (B, σ).

The time allocated to human capital accumulation on a BGP is determined by combining the third and fifth equations in (12) to get

$$v^{bg} = \left[1 + \frac{1-\sigma}{\omega\eta}\left(\frac{r^{bg}+\delta_H}{g+\delta_H} - \eta\right)\right]^{-1}. \tag{14}$$

Since $r^{bg} > g$ and $\eta < 1$, the second term in brackets is positive and $v^{bg} \in (0,1)$. Notice that v^{bg} is increasing in the subsidy σ, with $v^{bg} \to 1$ as $\sigma \to 1$. It is also increasing in ω and η, the elasticities for human capital in the two technologies. A higher value for ω increases the sensitivity of the wage rate $w(H)$ to private human capital, increasing the incentives to invest, while a higher value for η reduces the force of diminishing returns in time allocated to human capital accumulation. Finally, v^{bg} is increasing δ_H and g, since along a BGP investment must offset depreciation and H must keep pace with A.

The ratio of technology to human capital is then determined by the fifth equation in (12),

$$z^{bg} \equiv \frac{a^{bg}}{h^{bg}} = \left(\frac{g+\delta_H}{\phi_0\left(v^{bg}\right)^{\eta}}\right)^{1/\zeta}. \tag{15}$$

Hence this ratio is decreasing in the subsidy σ. Note that v^{bg} and z^{bg} do not depend on the barrier B.

The last equation in (12) then implies that a BGP level for the relative technology a^{bg}, if any exists, satisfies the quadratic

$$a^{bg}\left(1-a^{bg}\right) = (g+\delta_A)\,\frac{B}{\psi_0} z^{bg}, \tag{16}$$

and the following result is immediate.

Proposition 1 *If*

$$(g+\delta_A)\,\frac{4z^{bg}}{\psi_0} < \frac{1}{B}, \tag{17}$$

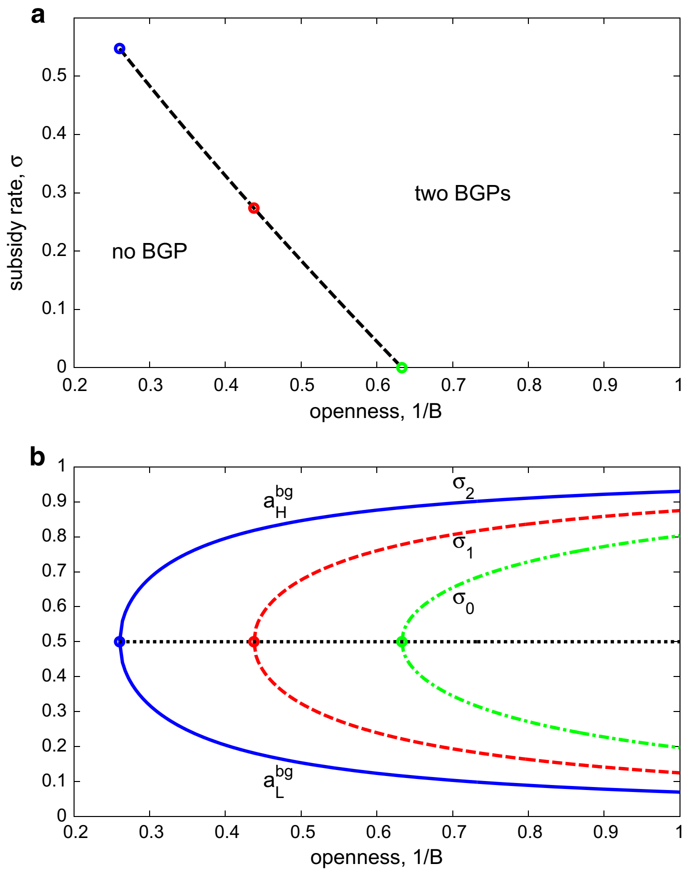

Fig. 4 **a** Policies for balanced growth. **b** Relative technology on BGP, $\sigma_0 < \sigma_1 < \sigma_2$

(16) *has two solutions. These solutions are symmetric around the value 1/2, and there is a BGP corresponding to each. If* (17) *holds with equality,* (16) *has one solution, and the unique BGP has* $a^{bg} = 1/2$. *If the inequality in* (17) *is reversed, no BGP exists.*

Thus, a necessary and sufficient for the existence of BGPs is that $1/B$, which measures "openness," exceed a threshold that depends—through z^{bg}—on σ. The threshold for $1/B$ increases as σ falls, as shown in Fig. 4a, creating two regions in the space of policy parameters. Above the threshold—for high values of $1/B$ and σ—the economy has two BGPs, and below the threshold there is no BGP.

If (17) holds, call the solutions a_L^{bg} and a_H^{bg}, with

$$0 < a_L^{bg} < \frac{1}{2} < a_H^{bg} < 1.$$

Figure 4b displays the solutions to (16) as functions of $1/B$, for three values of the subsidy σ and fixed values for the model parameters. For fixed σ, an increase in $1/B$ moves both solutions away from the value 1/2. For fixed $1/B$, an increase in σ has the same effect. For $(1/B, \sigma)$ below the threshold in Fig. 4a, no BGP exists.

Notice that the higher solution a_H^{bg} has the expected comparative statics—it increases with openness $1/B$ and with the subsidy σ—while the lower solution a_L^{bg} has the opposite pattern. As we will see below, a_H^{bg} is stable and a_L^{bg} is not. Therefore, since $a_H^{bg} \in [1/2,\ 1]$, this model produces BGP productivity (and income) ratios of no more than two across growing economies. Poor economies cannot grow along BGPs, in parallel with richer ones.[11]

In addition, since v^{bg} and z^{bg} do not vary with B, and only the higher solutions to (16) are stable, looking across economies with similar education policies σ, and all on BGPs, those with higher barriers B lag farther behind the frontier, but in all other respects are similar. Stated a little differently, an economy with a higher barrier looks like its neighbor with a lower one, but with a time lag.

3.2 Steady states for economies that stagnate

While BGPs exist only for policy parameters (σ, B) where (17) holds, every economy has a stagnation SS. At this SS, the technology level, capital stocks, factor returns, consumption, and time allocation are constant. Let A^{st}, H^{st}, K^{st}, and so on denote these levels.

Since consumption is constant, the SS interest rate is equal to the rate of time preference,

$$r^{st} = R^{st} - \delta_K = \rho.$$

Thus, the model predicts that the interest rate—the rate of return on capital—is lower in a stagnating economy than in one that is on its BGP. This is consistent with the evidence in Caselli and Feyrer (2007), who find that the return to capital is slightly lower in poor countries than in rich ones, 6.9 % rather than 8.4 %.

The input ratio κ^{st} and the return to effective labor $\hat{w}^{st}$ are then determined as before. The steady state time allocation and input ratio, the analogs to (14) and (15), are

$$v^{st} = \left[1 + \frac{1-\sigma}{\eta\omega}\left(\frac{r^{st}+\delta_H}{\delta_H} - \eta\right)\right]^{-1}, \tag{18}$$

$$z^{st} \equiv \frac{A^{st}}{H^{st}} = \left(\frac{\delta_H}{\phi_0\,(v^{st})^{\eta}}\right)^{1/\zeta}, \tag{19}$$

so they are like those on a BGP except that the interest rate is r^{st} and there is no growth.

Notice that for two economies with the same education subsidy σ , more time is allocated to human capital accumulation in the growing economy, $v^{bg} > v^{st}$, if and only if

$$\frac{r^{bg}+\delta_H}{g+\delta_H} < \frac{r^{st}+\delta_H}{\delta_H},$$

or

$$(\theta - 1)\,\delta_H < \rho.$$

If θ is sufficiently large, the low willingness to substitute intertemporally discourages investment in the growing economy. For $\theta = 2$ and $\delta_H = \rho$, as will be assumed in the simulations here, the steady state time allocations are the same, $v^{bg} = v^{st}$. In this case the growing economy has a higher ratio of technology to human capital,

$$\frac{z^{bg}}{z^{st}} = \left(\frac{g+\delta_H}{\delta_H}\right)^{1/\zeta} > 1.$$

[11] Other factors, like taxes on labor income (which reduce labor supply) can also increase the spread in incomes across growing economies. See Prescott (2002, 2004) and Ragan (2013) for models of this type.

4 Transitional dynamics

With three state variables, A, H, and K, and two costates, Λ_H and Λ_K (or their normalized counterparts), the transitional dynamics are complicated. The more interesting interactions involve technology and human capital, with physical capital playing a less important role. Thus, for simplicity we will drop physical capital to study transitions. Then $\alpha = 0$, $\hat{w} = 1$, and $R = 0$, and $C^{-\theta}$ takes the place of Λ_k.

Recall from Fig. 4a that if the policy $(1/B, \sigma)$ lies above the threshold, condition (17) holds, and the economy has two BGPs and one SS. As we will see below, in this case the SS and one of the BGPs are stable, while the other BGP, which separates them, is unstable. In this case the economy converges to the stable BGP if the initial conditions (a_0, h_0) lie above a certain threshold in the state space, and converges to the SS if the initial conditions lie below the threshold. If the policy $(1/B, \sigma)$ lies below the threshold in Fig. 4a, the economy has no BGP, and it converges to the SS for all initial conditions.

4.1 Catching up: transitions to a BGP

Suppose that $(1/B, \sigma)$ lies above the threshold in Fig. 4a, and consider the local stability of the two BGPs. Since $\overline{h} = h$, and all output is consumed,

$$c = (1 - v)\, a^{\beta} h^{1-\beta}.$$

Using $c^{-\theta}$ for λ_k, the first equation in (12) implies that the time allocation on either BGP satisfies

$$v^{\eta-1} (1 - v)^{\theta} = \frac{1 - \sigma}{\eta\phi_0} a^{\Delta} h^{-(\Delta+\theta)} \lambda_h^{-1}, \tag{20}$$

where $\Delta \equiv \beta (1 - \theta) - \zeta$. The transitional dynamics are then described by the three equations in (12), for $\dot{\lambda}_h/\lambda_h$, $\dot{h}/h$, and $\dot{a}/a$.

Recall that the normalized variables are constant along either BGP. This system of three equations can be linearized around the each of the steady states for the normalized variables, and the characteristic roots determine the local stability of each. In all of the simulations reported here, all of the roots are real, exactly two roots are negative at a_H^{bg}, and exactly one root is negative root at a_L^{bg}. Moreover, extensive computations suggest that this configuration for the roots holds for all reasonable parameter values.[12]

With this pattern for the roots, the higher steady state, a_H^{bg}, is locally stable: for any pair of initial conditions (a_0, h_0) in the neighborhood of $\left(a_H^{bg}, h_H^{bg}\right)$, there exists a unique initial condition λ_{h0} for the costate with the property that the system converges asymptotically. The lower steady state, a_L^{bg}, is unstable in the sense that there is only a one-dimensional manifold of initial conditions (a_0, h_0) in the neighborhood of $\left(a_L^{bg}, h_L^{bg}\right)$ for which the system converges. This manifold is the boundary of the basin of attraction for $\left(a_H^{bg}, h_H^{bg}\right)$. Above it the economy converges to the stable BGP, and below it the economy converges to the stagnation SS. Since the boundary between the two regions involves the normalized variables, it depends on W, as well as (A, H).[13]

[12] The Appendix contains a further analysis of the characteristic equation, and provides an example with complex roots.

[13] The boundary between the two regions is computed by perturbing around the point $\left(a_L^{bg}, h_L^{bg}\right)$ using the eigenvector associated with the single negative root, and running the ODEs backward. Transitional dynamics

4.2 Falling behind: transitions to the SS

Our interest here is in transitions to the stagnation SS from above. A transition of this sort would be observed in a country that had enjoyed growth above the stagnation level, by reducing its barrier or increasing subsidy to human capital accumulation, and then reversed those policies. This is not merely a theoretical possibility. As shown in Fig. 3, a number of African countries had negative growth in per capita GDP over the period 1960–2008, and even more had shorter episodes—one or two decades—of negative growth.

The time allocation during the transition to the SS satisfies the analog to (20),

$$v^{\eta-1}(1-v)^{\theta} = \frac{1-\sigma}{\eta\phi_0} A^{\Delta} H^{-(\Delta+\theta)} \Lambda_H^{-1}, \tag{21}$$

where Δ is as before, and the law of motion for the costate is

$$\frac{\dot{\Lambda}_H}{\Lambda_H} = \rho + \delta_H - \phi_0 \eta v^{\eta} \left(\frac{A}{H}\right)^{\zeta} \left(1 + \frac{\omega}{1-\sigma}\frac{1-v}{v}\right). \tag{22}$$

The transitional dynamics are described by these two equations and the laws of motion for W, A, H in (1)–(3).

A procedure for calculating transition paths is described in the Appendix. Note that an economy converging to the SS stagnates in the long run, but it may grow—slowly—for a while. Its rate of technology adoption eventually falls short of g, however, and after that the decay phase begins.

5 Calibration

The model parameters are the long run growth rate g; the preference parameters (ρ, θ); and the technology parameters $(\eta, \zeta, \phi_0, \delta_H)$ for human capital accumulation, (β, ω) for effective labor, and (ψ_0, δ_A) for technology diffusion. Baseline values for these parameters are described below. Experiments with alternative values are also conducted, to assess the sensitivity of the results.

The growth rate is $g = 0.019$, which is the rate of growth of per capita GDP in the US over the period 1870–2003.

The preference parameters are $\rho = 0.03$ and $\theta = 2$, which are within the range that is standard in the macro literature.

The depreciation rate for human capital is $\delta_H = 0.03$, which is close to the value ($\delta_H = 0.037$) estimated in Heckman (1976), and the same rate is used for technology, $\delta_A = 0.03$. For these parameters the time allocation is the same along the BGP and in the stagnation steady state, $v^{bg} = v^{st}$, for economies with the same subsidy σ.

For human capital accumulation, the elasticity η with respect to own human capital input vH is $\eta = 0.50$, which is close to the value ($\eta = 0.52$) estimated in Heckman (1976). The division of the remaining weight $1-\eta$ between A and $\overline{H}$ affects the speed of convergence, with more weight on A producing faster transitions. The baseline simulations use $\zeta = 1-\eta = 0.50$, so average human capital plays no role. The effect of a positive elasticity with respect to $\overline{H}$ is studied as part of the sensitivity analysis.

Footnote 13 continued
to the stable BGP are computed by perturbing around the point $\left(a_H^{bg}, h_H^{bg}\right)$ and running the ODEs backward. The perturbation can use any linear combination of the eigenvectors associated with the two negative roots, giving a two-dimensional set of allowable perturbations.

For goods production, the elasticity ω can be calibrated by using information from a Mincer regression. Consider families (dynasties) that invest more or less than the economy-wide average in human capital accumulation. If family j allocates the share of time v_j instead of v^{bg} along the BGP, then its human capital at any date t is

$$H_{jt} = \overline{H}_t \left(\frac{v_j}{v^{bg}}\right)^{\eta/1-\eta}.$$

From (6), the wage rates in a cross-section of such families at t then satisfy

$$\ln w_{jt} = \frac{\omega\eta}{1-\eta} \ln v_j + \text{constant}_t. \tag{23}$$

We can follow the procedure in Hall and Jones (1999) to obtain an empirical counterpart. Those authors use information from wage regressions in Psacharopoulos (1994) to construct

$$\ln w(e) = \Phi(e),$$

where $\Phi(e)$ is concave and piecewise linear, with breaks at $e = 4$ and $e = 8$, and slopes 13.4, 10.1, and 6.8 %. Suppose that an individual's time in market activities (education plus work) is about 60 years. Then $v = e/60$ is the fraction of the working lifetime that is devoted to education. In this case the function $\Phi(e)$ calculated by Hall and Jones is well approximated by the function in (23) with $\omega\eta/(1-\eta) = 0.55$, so $\eta = 0.50$ implies $\omega = 0.55$.

Here, as before, the division of $1-\omega$ between A and $\overline{H}$ affects the speed of convergence. The baseline simulations use $\beta = 1-\omega = 0.45$, so average human capital plays no role. The effect of a positive elasticity with respect to $\overline{H}$ is studied as part of the sensitivity analysis.

The constants ϕ_0, ψ_0 involve units for A and H, so one can be fixed arbitrarily. Here ϕ_0 is chosen as follows. Define a "frontier" economy as one with no barrier, $B_F = 1$, and σ_F set so that the time allocation v_F^{bg} on the BGP coincides with the one that solves a social planner's (welfare maximization) problem. Thus the barrier is optimal, and the education subsidy is optimal in the long run.[14] The parameter ϕ_0 is chosen so that $a_F^{bg}/h_F^{bg} = z_F^{bg} = 1$ in this frontier economy. From (15), this requires

$$\phi_0 = (g + \delta_H)\left(v_F^{bg}\right)^{-\eta}. \tag{24}$$

The constant ψ_0 affects both the speed of convergence and the maximum difference in income levels along BGPs. In particular, recall from (16) that regardless of the parameterization, the relative technology on a stable BGP is bounded above and below, $1/2 < a_H^{bg} < 1$. The choice of ψ_0 lowers the upper bound below its theoretical limit of unity, however, since even a frontier economy has a (small) gap between its local technology and the world technology. Let $a_F < 1$ denote the relative technology for the frontier economy. Then the maximum income ratio for economies on BGPs is $a_F/(1/2) = 2a_F$. Figure 2 shows a number of countries near the 45° line with incomes only slightly above half of the US level, suggesting a_F is not far below unity. Here we set $a_F = 0.94$, which implies $\psi_0 = 0.869$ and $\phi_0 = 0.121$. The subsidy and time allocation in the frontier economy are then $\sigma_F = 0.066$ and $v_F = 0.164$. Experiments with higher and lower values for a_F are included in the sensitivity analysis.

To summarize, the baseline parameters are

$$\begin{aligned}
&g = 0.019, \ \theta = 2.0, \quad \rho = 0.03, \\
&\eta = 0.50, \quad \zeta = 0.50, \ \delta_H = 0.03, \ z_F = 1, \\
&\omega = 0.55, \quad \beta = 0.45, \ \delta_A = 0.03, \ a_F = 0.94,
\end{aligned}$$

[14] The Social Planner's problem is described in the Online Appendix. For initial conditions off the BGP, a fully optimal policy requires a subsidy that varies along the transition path.

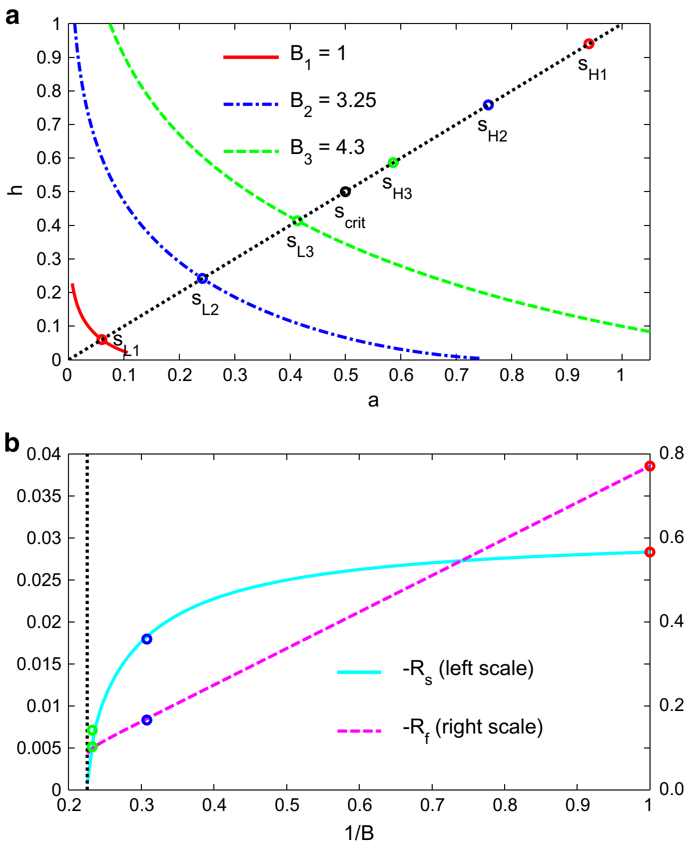

Fig. 5 **a** Basins of attraction for BGPs. **b** "Fast" and "slow" roots

which implies

$$\phi_0 = 0.121, \qquad \psi_0 = 0.869.$$

6 Baseline simulations

In these simulations the baseline model parameters are used and the subsidy to human capital is fixed at its level in the frontier economy, $\sigma_F = 0.066$.

6.1 Basins of attraction, phase diagrams

Figure 5 shows the effects of varying the barrier B. Figure 5a shows the (normalized) BGPs, $s_{Ji} = \left(a_{Ji}^{bg}, h_{Ji}^{bg}\right)$, $J = H, L$, $i = 1, 2, 3$, and basins of attraction for three barriers, $B_1 < B_2 < B_3$. Since the subsidy σ is the same for all three economies, the input ratios a_{Ji}^{bg}/h_{Ji}^{bg} on the BGP are also the same, and the normalized steady states lie on a ray from the origin. The normalization for ϕ_0 used here implies that ray is the 45° line. Since $\sigma = \sigma_F$ and

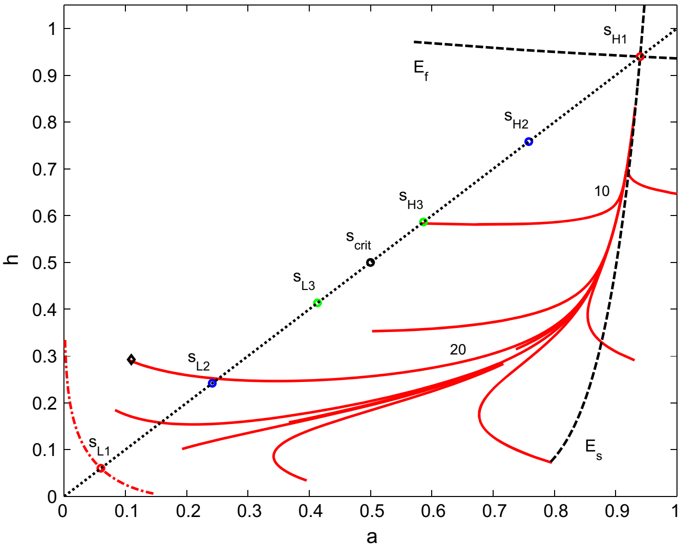

Fig. 6 Phase diagram, B = 1

$B_1 = 1$, the point s_{H1} represents the frontier economy, $a_{H1}^{bg} = a_F$. For $B > B_{crit} \approx 4.43$ the system does not have a BGP.

For each B_i, Fig. 5a also displays the threshold that separates the state space into two regions. For initial conditions above the threshold an economy with barrier B_i converges to the (stable) BGP described by s_{Hi}, and for initial conditions below the threshold it converges to its stagnation SS. For initial conditions exactly on the threshold it converges to the (unstable) BGP described by s_{Li}.

As in Fig. 4, increasing B shifts the threshold upward, $s_{L1} < s_{L2} < s_{L3}$, shrinking the set of initial conditions that produce balanced growth. Increasing B also shifts the normalized levels for the stable BGP downward, $s_{H1} > s_{H2} > s_{H3}$. Thus, comparing across economies on BGPs for different barriers, those with higher barriers lag farther behind the frontier economy.

The barrier B is also important for the speed of convergence to the stable BGP. Figure 5b displays the values for the two negative roots, which govern that speed, as functions of $1/B$. Call the roots "fast" and "slow," with $-R_f > -R_s > 0$. Greater openness—a higher value for $1/B$—increases both roots, speeding up transitions. The fast root is approximately linear in $1/B$, implying that a low barrier will be needed to construct a growth miracle. The slow root is a concave function of $1/B$, converging to zero as $1/B$ declines to the threshold where a BGP ceases to exist.

Figure 6 shows the phase diagram for $B = 1$. The solid lines are transition paths to the stable BGP, the broken line near the origin is the threshold from Fig. 5a, and the dashed lines are the eigenvectors, E_f and E_s, associated with the negative roots. (The latter are linear in the log space.) The transition paths from the left are rather flat, indicating that the technology level a adjusts more rapidly than human capital h. Increasing B makes these paths steeper, as Fig. 5b suggests. Roughly speaking, R_f governs the adjustment of a and R_s the adjustment of h, and—over most of the range—changing the barrier has a relatively larger impact on the former.

6.2 Transition paths: fast growth and miracles

First consider an economy that is on a BGP with a high barrier and lowers that barrier. In particular, consider the transition for an economy that lowers its barrier from B_3 to B_1, so its initial and final conditions are s_{H3} and s_{H1} in Figs. 5 and 6. The ratio $z^{bg} = a^{bg}/h^{bg}$ is the same on both BGPs, and both state variables start at about $0.59/0.94 \approx 63\,\%$ of their final levels.

The transition path in a, h-space is one of the trajectories displayed in Fig. 6. Over the first decade of the transition the normalized technology a grows rapidly while the normalized human capital stock h grows very slowly. The economy's position after 10 years is as indicated in the figure. Thereafter the relative technology keeps pace with the frontier, and relative human capital slowly catches up over the next decades.

Figure 7 displays time plots for the first 50 years after the policy change. Figure 7a shows the share of time v allocated to human capital accumulation, which falls sharply immediately after the policy change. Two factors are at work. First, consumption smoothing provides a direct incentive to shift the time allocation toward goods production. In addition, since the local technology is an important input into human capital accumulation, there is an incentive to delay the investment of time in that activity until after the complementary input has increased. In a model with endogenous leisure, the total time allocated to market activities—work and human capital accumulation—presumably would rise, mitigating or even eliminating the fall in v. This in turn would imply more rapid human capital growth during the early years of the transition.

Figure 7b shows the wage rate w and output $y = (1 - v)\,w$ relative to the frontier economy, and Fig. 7c shows their growth rates. Both display rapid growth in the early years of the transition, but the growth rates taper off rather quickly, falling back toward the steady state level of 1.9 %. Output jumps on impact, as time spent working jumps. Because of this initial jump, output growth in the early years of the transition is slower than wage growth. After the first decade they are about equal, and both fall back toward the BGP level.

Although the economy in Fig. 7 enjoys a short period of rapid growth, it is not a miracle by the criterion in Sect. 1, which required an average growth rate of at least 5 % over 20 years. By that criterion, income must increase by at least a factor of $e^{(.05)(20)} = 2.57$ over 20 years. But the initial condition for an economy on a BGP is at least half of income in the frontier economy, so its income cannot double. Thus, growth miracles must have initial conditions below those possible on any BGP. What sort of initial conditions do the data suggest?

The miracles in Germany and Italy are probably explained in large part as recovery in the physical capital stock after the destruction in World War II, a quite different mechanism from the one proposed here. For the others, initial per capita incomes were 10 % (Taiwan and S. Korea) and roughly 23 % (Japan, Hong Kong, Singapore, Greece, Portugal, Spain, Puerto Rico, Israel) of the US level when their periods of rapid growth began. For the benchmark model parameters and the policy parameters $B = 1$ and $\sigma = \sigma_F$, all would lie inside the trapping region for balanced growth.[15]

Here we will look at the transition for an economy whose initial income is 20 % of the level in the frontier economy. The initial technology level and human capital stock are chosen to be consistent with the description of a stagnation SS, $(a_0, h_0) = (0.11,\ 0.29)$, with

[15] Nevertheless, since growth is very slow near this boundary, the model has difficulty fitting Taiwan and S. Korea. China's per capita income was about 6.5 % of the US level when its period of rapid growth began, which puts slightly outside the trapping region in Fig. 6.

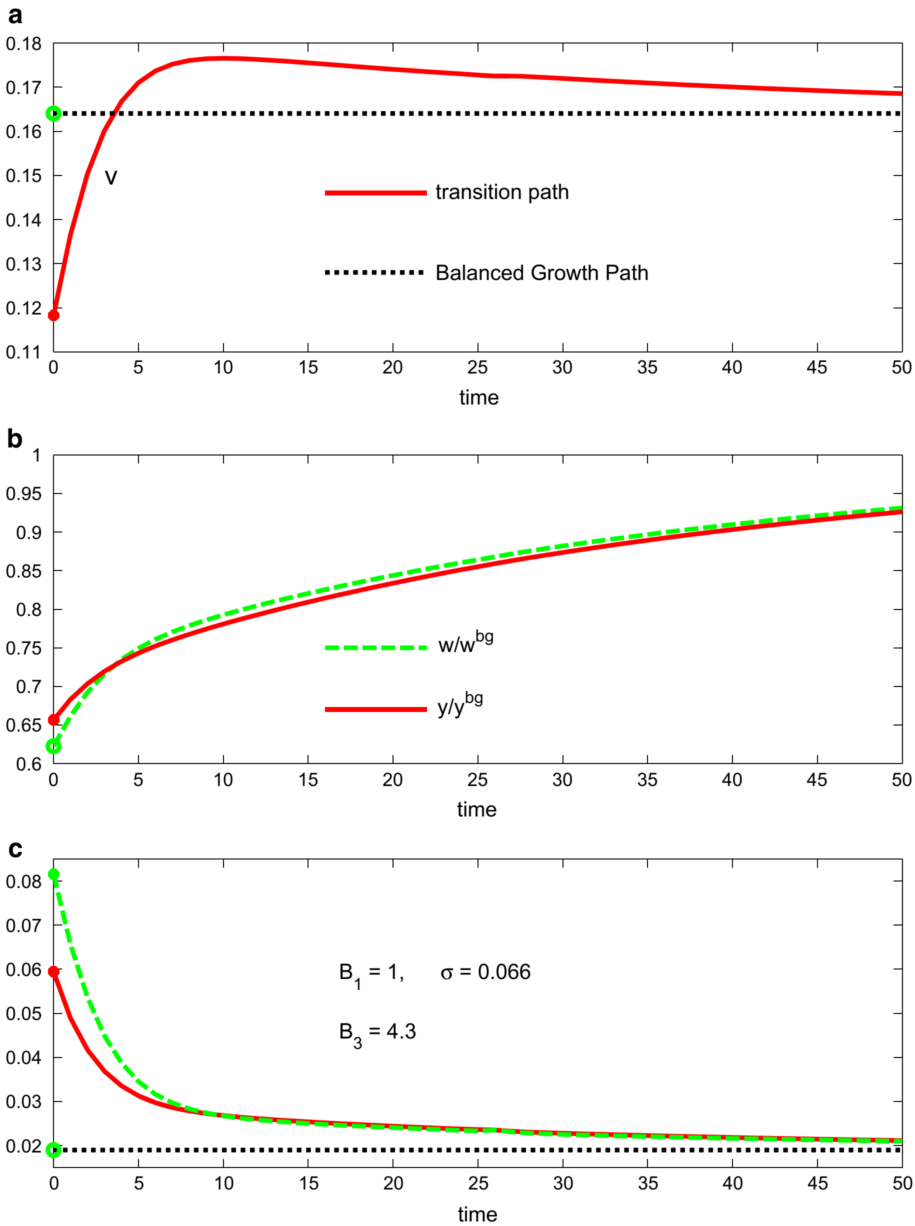

Fig. 7 **a** Time allocation for catch-up growth. **b** Wage and output, relative to new BGP. **c** Wage and output growth rates

$a_0/h_0 = z^{st}$. This transition path is also shown in Fig. 6, where it is the path starting at a small diamond.[16]

16 The human capital subsidy is assumed to be σ_F throughout, but using $\sigma = 0$ in the initial condition does not change the results much.

During the first phase of the transition, which lasts for about 20 years, the local technology relative to the world frontier, $a = A/W$, grows rapidly, rising from 0.11 to 0.67. Local human capital $h = H/W$ declines slightly and then recovers to almost exactly its original level, 0.29, implying that human capital has grown at about 1.9 % per year over those two decades. The economy's position after 20 years is as indicated in Fig. 6.

Figure 8 shows time plots for the first 100 years of the transition.[17] The patterns are qualitatively similar to those in Fig. 7. The initial swing in the time allocated to human capital accumulation, displayed in Fig. 8a, is more pronounced here. It is this swing that produces the initial decline in relative human capital. The wage and output and their growth rates are shown in Fig. 8b, c. Wage growth exceeds 5 % over the first two decades, so this economy qualifies as a "miracle" if wage growth is used as the measure. Output growth is a little lower, because part is absorbed in the jump induced by the initial jump in time allocation. As noted above, making leisure endogenous would presumably increase output growth during this initial phase of the transition. The growth rates for both the wage and output remain above 3 % for two more decades, and then decline gradually back toward the BGP level of 1.9 %.

6.3 Policies and starting points

Both the policy in place and the initial conditions affect the speed of the transition, as the next two experiments illustrate.

Figure 9 displays the transitions for two different policies for economies with the same initial conditions. The initial condition for both is the BGP position for a modest barrier, $B_C = 1.35$, and no subsidy to human capital accumulation, $\sigma_C = 0$. Policy D (solid line) lowers the barrier to $B_D = 1$, with no change in the subsidy, while Policy E (broken line) increases the subsidy to the level in the frontier economy, $\sigma_E = 0.066$, with no change in the barrier. The two policies are constructed so that output (consumption) on the BGP is the same under both.

The sources of income growth are very different, however. Policy D exploits technology inflows, a costless and rapidly acquired input, while Policy E relies on investment in local human capital, a slow and costly process. Thus, the economy adopting Policy D enjoys a much more rapid transition to the new BGP. Figure 9a shows the two transitions in the state space. Under Policy D, the local technology increases very quickly, and after 5 years it has grown to almost its final relative position. Under Policy E, the state variables show little change after 5 years.

Figure 9b shows the time allocation. Under Policy D the BGP time allocation is unaltered, since the subsidy is unchanged. As in the previous experiments, the incentive to smooth consumption leads to an initial swing, although here the swing is small. Consequently human capital grows mainly because of improvements in the local technology. This part of the adjustment is slow, requiring several decades. Under Policy E, the increase in the subsidy induces an immediate jump in the time allocated to human capital accumulation, which overshoots its new (higher) BGP level.

Figure 9c–f show output, the wage, and their growth rates. Under Policy D the technology inflow produces an initial phase of very rapid growth. Output jumps slightly on impact, because of the jump in the time allocation, and as a result subsequent output growth is slightly lower than wage growth. Under Policy E output declines on impact because of the shift in time allocation, attaining its previous (relative) position only after about two decades.

[17] The first 42 years are computed exactly, and the remainder with a log-linear approximation.

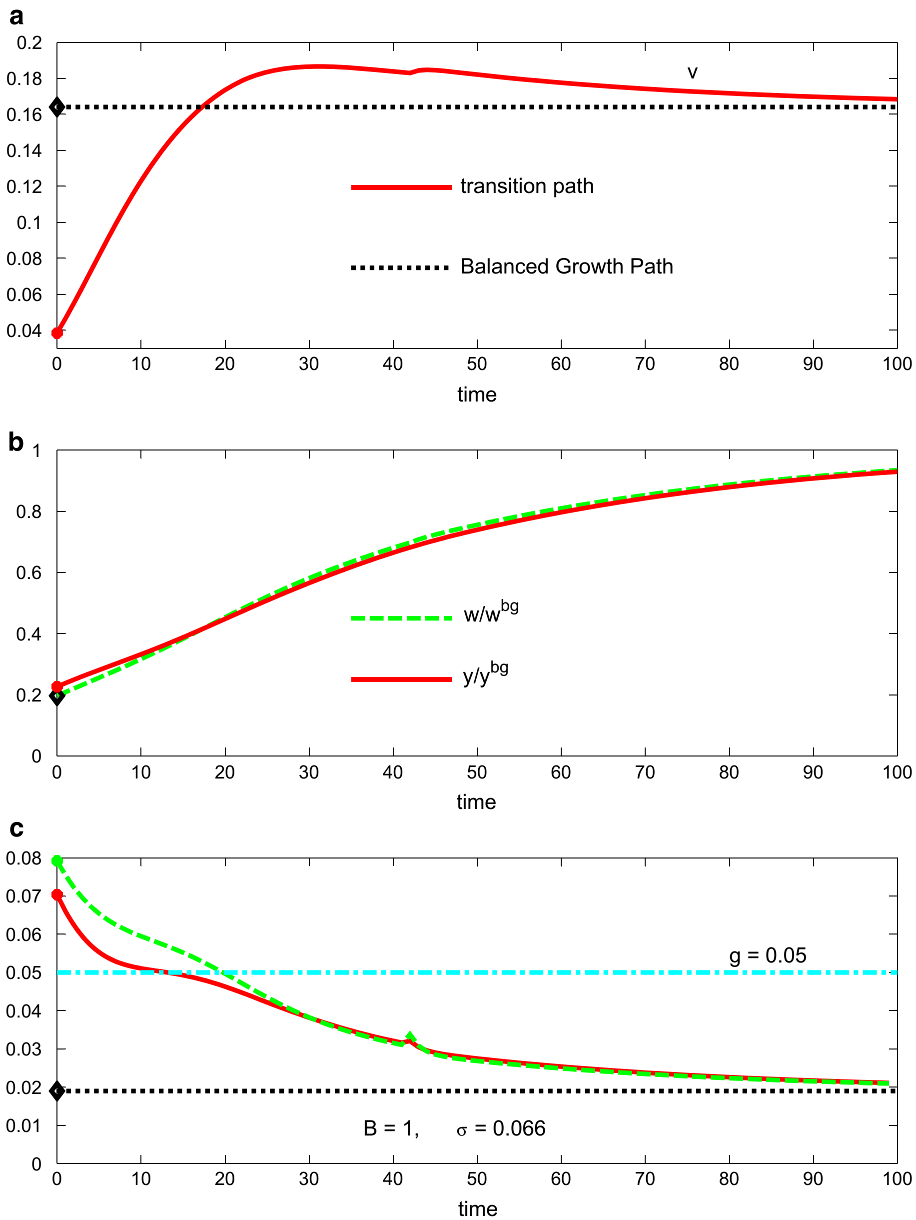

Fig. 8 **a** Time allocation for a growth miracle. **b** Wage and output, relative to new BGP. **c** Growth rates of wage and output

The growth rates for both output and the wage increase only slightly, from 1.9 to 2.0 %, during the initial years of the transition.

By construction the two policies lead to the same BGP level for y, as shown in Fig. 9c. The BGP wage rate, in Fig. 9e, is somewhat higher under Policy E, but less time is allocated

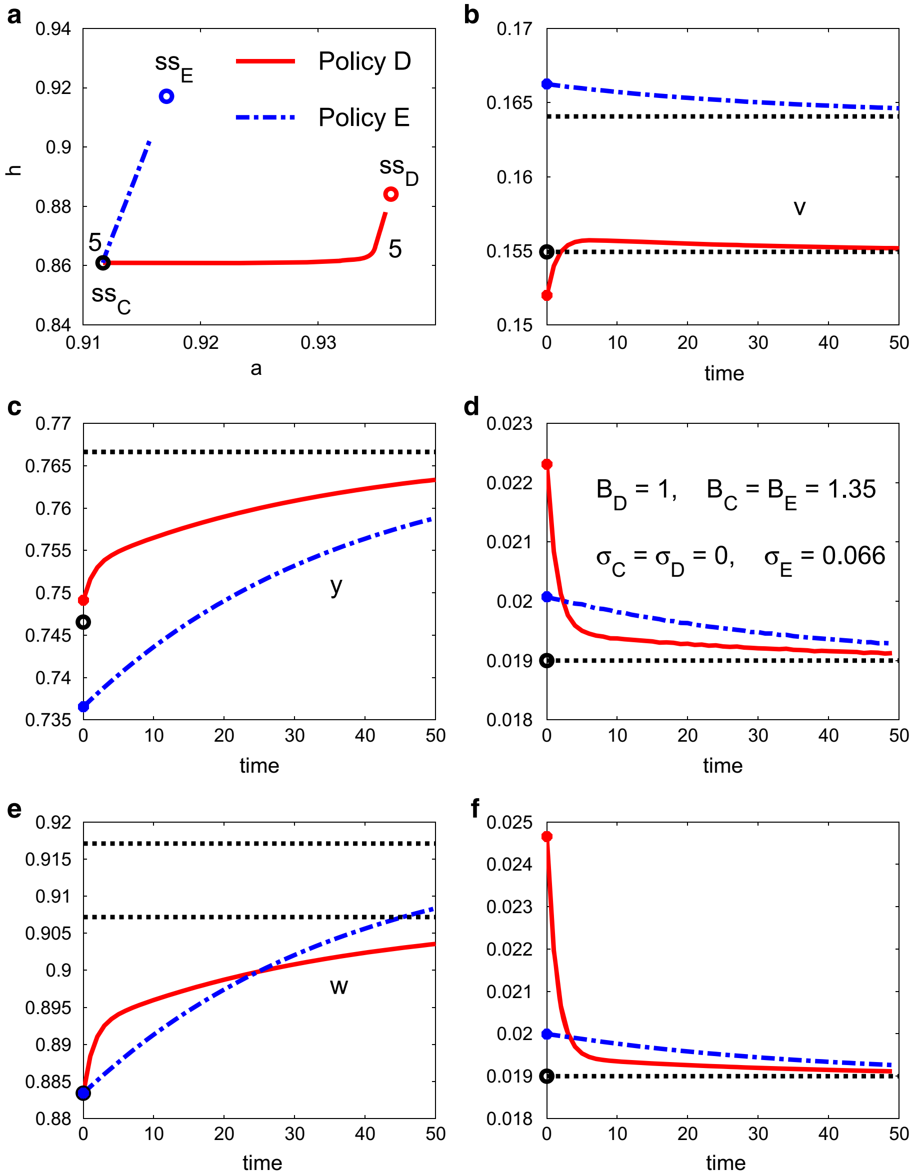

Fig. 9 **a** Two policies. **b** Time allocation. **c** Output. **d** Output growth. **e** Wage rate. **f** Wage growth

to work. Clearly welfare during the transition is higher under Policy D, since output is higher throughout.

Next consider the role of initial conditions. Figure 10 shows the transitions for two economies with different initial values for (a, h) but the same initial output level. The initial conditions are the terminal steady states in Fig. 9. Economy D (the solid line) starts on the BGP for $B_D = 1$, $\sigma_D = 0$, while Economy E (the dashed line) starts on the BGP for the policy $B_E = 1.35$, $\sigma_E = 0.066$. Economy D increases its subsidy to $\sigma_F = 0.066$, while

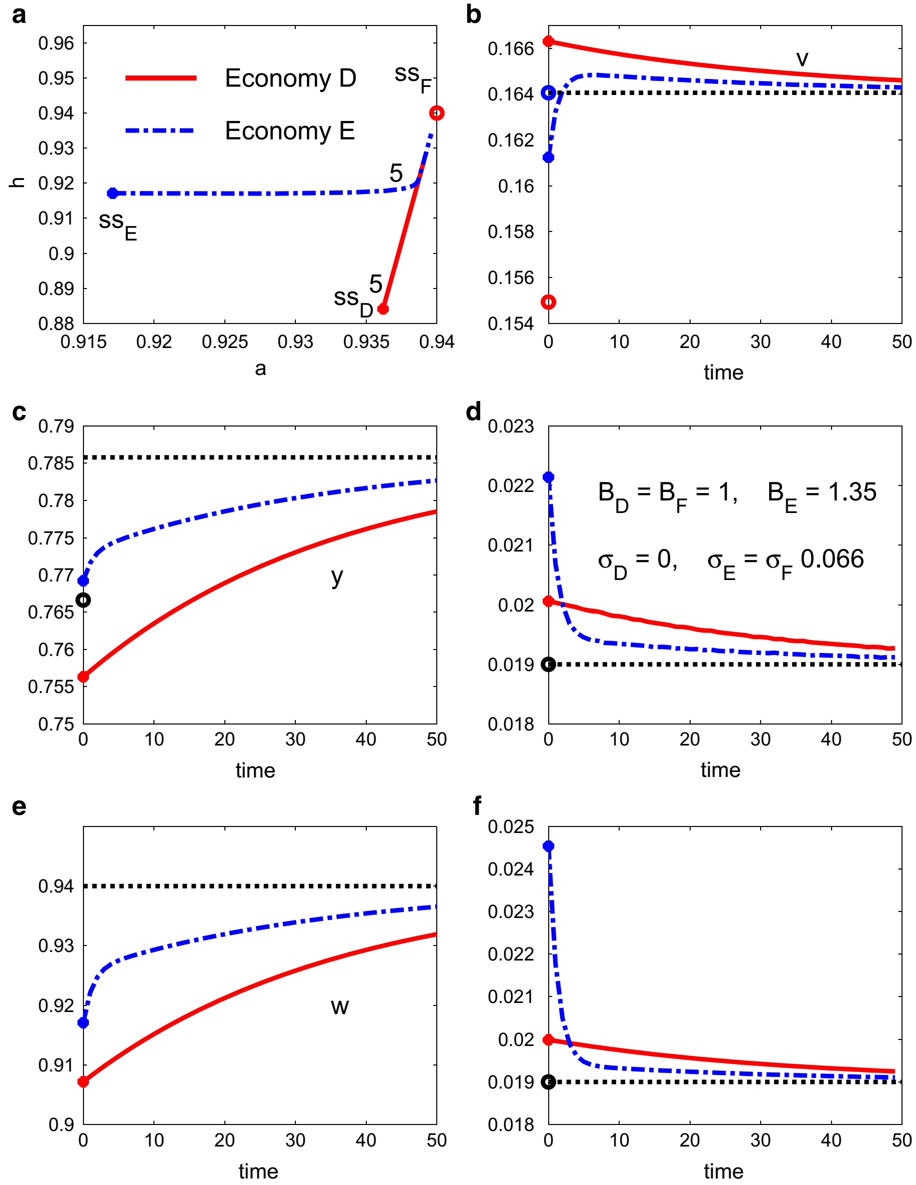

Fig. 10 **a** Two economies. **b** Time allocation. **c** Output. **d** Output growth. **e** Wage rate. **f** Wage growth

Economy E lowers its barrier to $B_F = 1$, so both are converging to the BGP of the frontier economy.

The transitions mirror those in the previous example. Here Economy E, which lowers its barrier, enjoys a rapid inflow of technology, shown in Fig. 10a. Its time allocation makes a modest swing, as in earlier examples where a barrier is lowered, but in the long run returns to its initial value. Output jumps up slightly on impact, and both output and the wage grow rapidly during the initial couple of years of the transition.

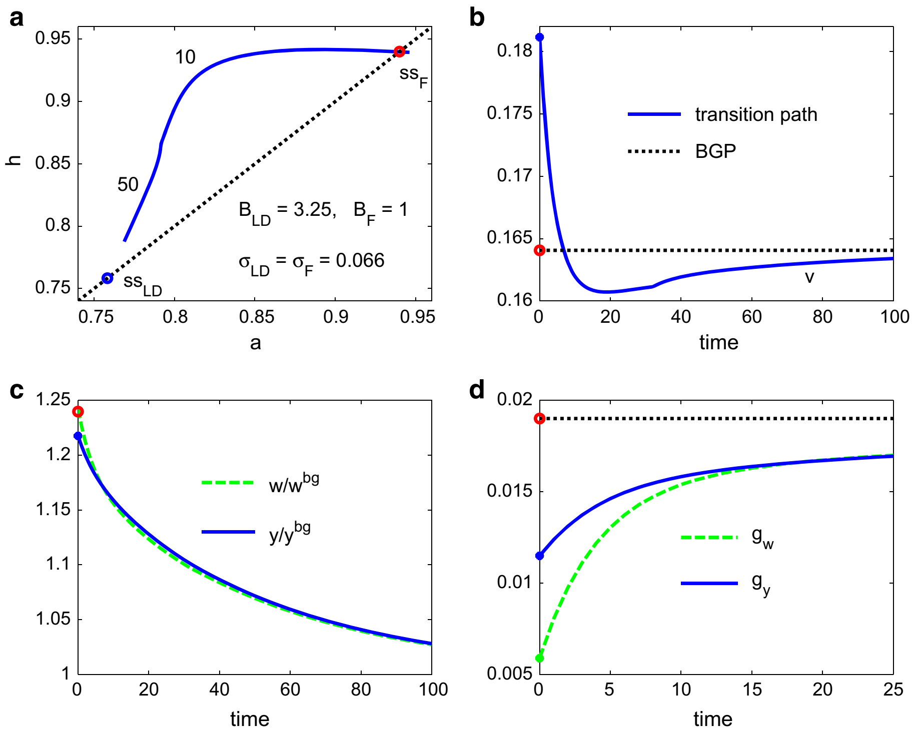

Fig. 11 **a** A lost decade. **b** Time allocation. **c** Wage, output relative to new BGP. **d** Wage, output growth

Economy D enjoys only sluggish growth. Its long-run share of time allocated to human capital accumulation increases because of the increase in the subsidy, as shown in Fig. 10b. The initial jump here is large, overshooting the ultimate level. Hence output suffers an initial fall, as shown in Fig. 10c. The output and wage lie for Economy D lie below those for Economy E during the whole transition.

6.4 Transitions: lost decades and disasters

Next consider economies that raise their barriers to technology inflows. Figure 11 displays the transition for an economy that raises its barrier from $B_F = 1$ to $B_{LD} = 3.25$. The result is a "lost decade" of growth, even though the economy is on a BGP before and after the transition. Figure 11a shows the transition in the state space, with the positions after 10 and 50 years indicated. Here, again, the local technology adjusts more rapidly than local human capital. As before, the effect of the policy change on time allocation, shown in Fig. 11b, exacerbates the slow adjustment of human capital. The pattern is a mirror image of the previous ones, and the logic is the same. Human capital accumulation is the only vehicle for smoothing consumption, and it uses the local technology as an input. Here, both forces lead the household to shift its time allocation toward accumulation just after the policy change. Figure 11c shows the transition paths for the wage rate and output relative to their new BGP levels. Output jumps down on impact, because of the shift in time allocation, and both decline steadily during the transition, adjusting by about 24 % in the long run. Figure 11d displays the growth rates of the wage and output, which fall from 1.9 to 0.6 % and 1.1 % respectively.

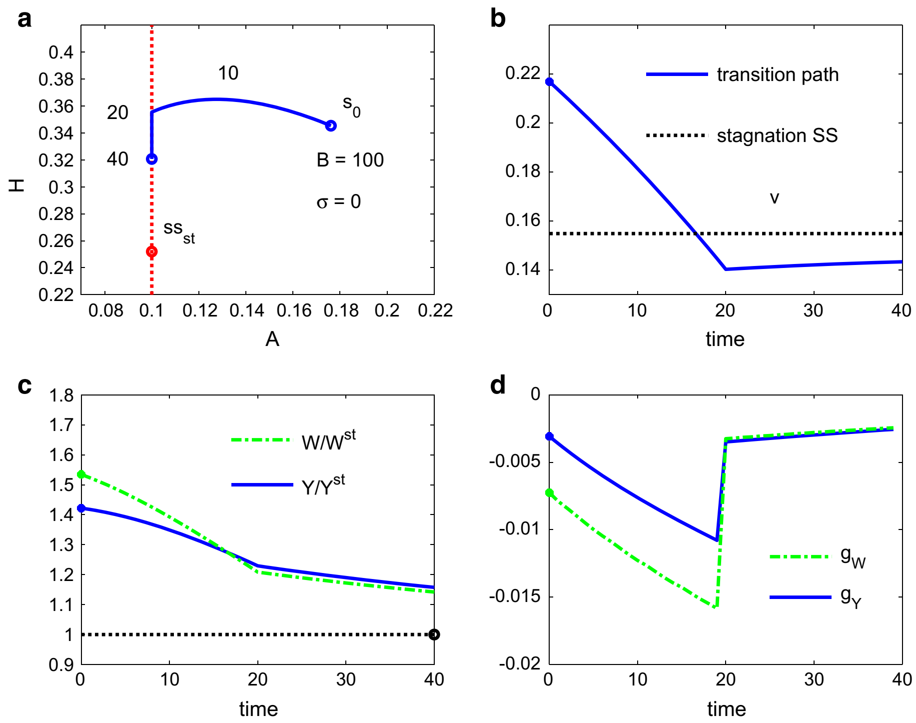

Fig. 12 **a** A growth disaster. **b** Time allocation. **c** Wage, output relative to SS_{st}. **d** Wage, output growth

Both gradually return to the BGP level, but the lost decade is never recovered: output on the new BGP is about 24 % below what it would have been on the old one.

Figure 12 shows a "growth disaster." The initial values A, H, and Y, for this economy (all in levels) are above their stagnation values, and Fig. 12 shows the transition to the stagnation SS. This is the scenario for an economy that in the past lowered its barrier to technology inflows and enjoyed a period of growth, arriving at state s_0, and now raises its barrier. Here the new barrier is set very high, so there is essentially no technology inflow and the local technology decays at the rate δ_A.

The transition in the state space, shown in Fig. 12a, is qualitatively similar to the previous one, with the local technology initially adjusting more rapidly than human capital. After 20 years the local technology has reached its lower bound, A^{st}, and thereafter only human capital adjusts. Here, as before, the household saves for its (dim) future the only way it can, by preserving its human capital. As shown in Fig. 12b, substantial time is allocated to human capital accumulation just after the policy change, reflecting the same two motives as before: consumption smoothing and utilizing the local technology before it decays further. The wage and output, shown in Fig. 12c, decline over the entire transition, although the rate of decline slows after the local technology hits its lower bound, as shown in Fig. 12d.

7 Sensitivity analysis

In this section we will study the sensitivity of the results to changes in the elasticities (β, ω) and (η, ζ) in the production functions, and the level parameter a_F, which determines the coefficient ψ_0 in the law of motion for technology inflows. The experiments assume no barrier, $B = 1$, and the subsidy to human capital accumulation is adjusted to the socially efficient level throughout, $\sigma = \sigma_F$. Thus, all of the experiments involve transitions to the relevant "frontier" BGP. In each case the economy starts on a BGP where output is 65 % of the frontier level, so the transitions are analogs to the one in Fig. 7. More detail about these experiments is provided in the Online Appendix.

The first experiments look at changes in the elasticities β and ω in the technology for goods production, with η, ζ, and a_F held at their baseline values. An important result here is that with the technology parameter β fixed, the division of the share $1 - \beta$ between private human capital and the external effect, between ω and $1 - \beta - \omega$, has virtually no effect if the subsidy adjusted to its efficient level. Although the decrease in ω reduces the incentive to invest, the increase in $1 - \beta - \omega$ implies a higher subsidy, which exactly restores the incentive.

Changes in β do have effects, however. Consider a lower value for β with an offsetting increase in ω. The private incentive to invest in human capital is then higher, so the time allocation is shifted toward accumulation. The growth rates for the wage and output are then lower during the first years of the transition, since a lower value for β means that the technology inflows induced by a reduction in B have a smaller immediate impact on goods production. If the lower value for β is instead offset by a change in $1-\omega-\beta$, with ω constant, the external effect increases. In this case the efficient subsidy σ_F rises, producing the same shift in the time allocation and the same effects on output and growth.

The second set of experiments looks at changes in the elasticities η and ζ in the technology for human capital accumulation, with β, ω, and a_F held at their baseline values. Because the term v^η appears in the investment technology, changes in η have a significant impact on the optimal time allocation. Nevertheless, changes in ζ and η produce only modest changes in the transition paths for output and the wage rate. This is not surprising: human capital accumulation affects output and wages slowly and gradually.

A reduction in ζ with η fixed shifts weight to the external effect. The efficient subsidy increases, but there is no effect through v^η, so the time allocation shifts slightly toward accumulation. The effect on the wage and output is extremely small in the early years of the transition, since changes in human capital accumulation show up slowly. A lower weight on technology slows the transition in the later years, however.

Next consider a change in ζ with an offsetting change in η. Since the external effect remains shut down, $1 - \eta - \zeta = 0$, the efficient subsidy σ_F changes very little. Nevertheless, increasing η increases the incentive to allocate time to accumulation. The transition paths for output and the wage change very little in the early years, but the transition is a little slower in the later years.

Finally, a reduction in η with ζ fixed shifts weight from private human capital to the external effect, and also reduces the elasticity in the time allocation term v^η. The first channel produces an increase in the efficient subsidy σ_F, but the second reduces the incentive, even with the higher subsidy, shifting the time allocation away from accumulation. The net effect on the transition paths and growth rates for the wage and output are small, however.

The last experiment looks at changes in a_F, which affect the coefficient ψ_0 in the law of motion for technology inflows. Changing a_F has a dramatic effect on the speed of adjustment. Recall that the maximum income ratio between countries on BGP is $2a_F$, so a large value for a_F is needed if this range is to be substantial. But the level parameter ψ_0 is proportional to $1/a_F\,(1 - a_F)$, so it is very sensitive to a_F when a_F is close to unity. Increasing a_F from 0.94 to 0.96 increases ψ_0 by about 47 %, which in the early years of the transition raises output growth rate by 2.0 % points and wage growth by 4.0 % points. The efficient subsidy falls, but time allocated to accumulation is not much changed. Reducing a_F from 0.94 to 0.91 produces similar effects in the opposite direction, making it very hard to generate a growth miracle. Thus, a_F must be close to its baseline value—or even a little higher—to produce growth episodes like those seen in the east Asian miracles.

8 Conclusions

The model developed here studies the interaction between technology, which flows in from abroad, and human capital, which is accumulated domestically, as an explanation of cross-country differences in income levels and growth rates. As argued in Sect. 1, several different types of evidence, from many countries and over long time periods, point to the importance of technology inflows. But technology, by itself, does not seem to be a sufficient explanation. In the model here local human capital is critical in letting countries effectively exploit technologies imported from abroad and in allowing that inflow to continue.

The model developed here has a number of empirical implications. First, as shown in Proposition 1, economies with sufficiently unfavorable policies—a combination of high barriers to technology inflows and low subsidies to human capital accumulation—have no BGP. Thus, the model implies that only middle and upper income economies can grow like the technology frontier over long periods. Low income economies can grow faster (if they have reduced their barriers) or slower (if they have raised them) or stagnate. This prediction is consistent with the empirical evidence: higher income countries grow at similar rates over long periods, while low income countries show more heterogeneity (in cross section) and variability (over time).

Second, unfavorable policies reduce the level of income and consumption on the BGP, and shrink the set of initial conditions for which an economy converges to that path. Thus, the model predicts that middle-income countries that grow with the frontier should have displayed slower growth during their transitional phases.

Third, growth miracles feature a modest period of very rapid TFP growth, accompanied by rapid growth in income and consumption. This initial phase is followed by a (longer) period during which human (and physical) capital are accumulated. Income growth declines toward the rate of frontier growth as the income level approaches the frontier level. Thus, the model has predictions for some features of the transition paths of growth 'miracles.'

Fourth, low income countries with higher human capital are better candidates to become growth miracles, since TFP and income grow more rapidly in such economies.

Fifth, the model suggests that policies stimulating technology transfer are far more effective in accelerating growth than policies stimulating human capital accumulation. Investments in human and physical capital respond to returns, and those returns are high when technology is growing rapidly. Thus, the empirical association between high investment rates and rapid growth is not causal: technology drives both.

The model studied here offers a view of how world technology affects the possibilities for less developed economies. The evidence discussed in Sect. 1, especially data on late bloomers and miracles, leaves little doubt that its effect is large, and the numerical exercises reported in Sect. 6 are consistent with that evidence. The model offers no explanation for the ultimate sources of technological innovation and, consequently, the paper has little to say about the determinants of the growth rate in developed economies. The sources of growth differ across countries according to their level of development relative to the world they are situated in, and quite different models are needed to study growth in advanced economies.

Appendix 1: Equilibrium conditions

The Hamiltonian for the household's problem in (9) is

$$\begin{aligned}\mathfrak{H} = \frac{C^{1-\theta}}{1-\theta} + \Lambda_H \left[\phi_0 (vH)^\eta A^\zeta \overline{H}^{1-\eta-\zeta} - \delta_H H\right] \\ + \Lambda_K \left[(1-v)\, \hat{w} A^\beta H^\omega \overline{H}^{1-\beta-\omega} + v\sigma \hat{w} A^\beta \overline{H}^{1-\beta} + (R-\delta_K) K - C - \tau\right],\end{aligned}$$

where to simplify the notation the subscript i's have been dropped. Taking the first order conditions for a maximum, using the equilibrium conditions $\overline{H} = H$ and $\tau = v\sigma\hat{w}A^\beta H^{1-\beta}$, and simplifying gives

$$\begin{aligned}\Lambda_H \phi_0 \eta v^{\eta-1} &= \Lambda_K (1-\sigma)\, \hat{w} \left(\frac{A}{H}\right)^{\beta-\zeta}, \\ C^{-\theta} &= \Lambda_K, \\ \frac{\dot{\Lambda}_H}{\Lambda_H} &= \rho + \delta_H - \phi_0 \eta v^\eta \left(\frac{A}{H}\right)^\zeta \left(1 + \frac{\omega}{1-\sigma}\frac{1-v}{v}\right), \\ \frac{\dot{\Lambda}_K}{\Lambda_K} &= \rho + \delta_K - R, \\ \frac{\dot{H}}{H} &= \phi_0 v^\eta \left(\frac{A}{H}\right)^\zeta - \delta_H, \\ \frac{\dot{K}}{K} &= \left(\frac{K}{L}\right)^{\alpha-1} - \frac{C}{K} - \delta_K, \end{aligned} \tag{25}$$

where $L = (1-v)\, A^\beta H^{1-\beta}$, and $\dot{A}/A$ is in (2). The transversality conditions are

$$\lim_{t\to\infty} e^{-\rho t} \Lambda_K(t) K(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} e^{-\rho t} \Lambda_H(t) H(t) = 0. \tag{26}$$

Equations (1), (2), (25) and (26) characterize the competitive equilibrium, given the policy parameters (B, σ) and initial values for the state variables (W, A, H, K).

The law of motion for A requires H/W and A/W to be constant along a BGP, so A and H must also grow at the rate g. Since the production functions for effective labor in (4) and for output have constant returns to scale, the factor inputs L and K also grow at the rate g, as do output Y and consumption C. Hence the factor returns R and $\hat{w}$ are constant on a BGP, and the costate variable Λ_K grows at the rate $-\theta g$. The costate Λ_H grows at the same rate as Λ_K.

The normalized conditions in (12) follow directly from (2) and (25).

Appendix 2: Linear approximations and stability

Define the constants

$$\chi \equiv \eta \frac{\omega}{1-\sigma}, \qquad \Pi_H \equiv \frac{r^{bg} + \delta_H}{g + \delta_H}, \qquad \Delta \equiv \beta(1-\theta) - \zeta,$$

$$\begin{aligned}
\Gamma_2 &\equiv \eta - \frac{\chi}{\eta v^{bg} + \chi\left(1 - v^{bg}\right)} < \eta, \\
&= \eta - \frac{\chi / v^{g}}{\eta + \chi\left(1/v^{bg} - 1\right)} \\
&= \eta - \frac{\chi + \Pi_H - \eta}{\Pi_H}, \\
\frac{1}{\Gamma_3} &\equiv \eta - 1 - \frac{\theta}{1/v^{bg} - 1} \\
&= \eta - 1 - \frac{\theta\chi}{\Pi_H - \eta} < 0,
\end{aligned}$$

and recall that

$$\frac{1}{v^{bg}} - 1 = \frac{1}{\chi}\left(\Pi_H - \eta\right) > 0.$$

In addition define the log deviations

$$x_1 \equiv \ln\left(a/a^{bg}\right), \qquad x_2 \equiv \ln\left(h/h^{bg}\right), \qquad x_3 \equiv \ln\left(\lambda_h/\lambda_h^{bg}\right),$$

Take a first-order approximation to (20) to get

$$\frac{v - v^{bg}}{v^{bg}} = \Gamma_3\left[\Delta x_1 - (\Delta + \theta)\, x_2 - x_3\right].$$

Then linearize the laws of motion for a, h, and λ_h in (12) to find that

$$\begin{aligned}
\dot{x}_1 &\approx (g + \delta_A)\left[-\frac{a_J^{bg}}{1 - a_J^{bg}} x_1 + x_2\right], \\
\dot{x}_2 &\approx (g + \delta_H)\left[\zeta\,(x_1 - x_2) + \eta\frac{v - v^{bg}}{v^{bg}}\right] \\
&= (g + \delta_H)\left\{\zeta\,(x_1 - x_2) + \eta\Gamma_3\left[\Delta\,(x_1 - x_2) - \theta x_2 - x_3\right]\right\}, \\
\dot{x}_3 &\approx -\left(r^{bg} + \delta_H\right)\left[\zeta\,(x_1 - x_2) + \Gamma_2\frac{v - v^{bg}}{v^{bg}}\right] \\
&= -\left(r^{bg} + \delta_H\right)\left\{\zeta\,(x_1 - x_2) + \Gamma_2\Gamma_3\left[\Delta\,(x_1 - x_2) - \theta x_2 - x_3\right]\right\}.
\end{aligned}$$

Hence

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} \approx \begin{pmatrix} -q_{1J} & q_2 & 0 \\ q_3 & -q_3 - \theta q_4 & -q_4 \\ -q_6 & q_6 + \theta q_5 & q_5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

where

$$\begin{aligned}
q_{1J} &= (g + \delta_A)\, a_J^{bg} / \left(1 - a_J^{bg}\right), \\
q_2 &= g + \delta_A, \\
q_3 &= (g + \delta_H)\,(\zeta + \eta\Gamma_3\Delta), \\
q_4 &= (g + \delta_H)\,\eta\Gamma_3, \\
q_5 &= \left(r^{bg} + \delta_H\right)\Gamma_2\Gamma_3, \\
q_6 &= \left(r^{bg} + \delta_H\right)(\zeta + \Gamma_2\Gamma_3\Delta).
\end{aligned}$$

The local stability of each steady state depends on the roots of the associated characteristic equation,

$$\begin{aligned}
0 &= \det \begin{pmatrix} -R - q_{1J} & q_2 & 0 \\ q_3 & -R - (q_3 + \theta q_4) & -q_4 \\ -q_6 & q_6 + \theta q_5 & -R + q_5 \end{pmatrix} \\
&= \det \begin{pmatrix} -(R + q_{1J}) & q_2 & 0 \\ q_3 & -(R + q_3) & -q_4 \\ -q_6 & q_6 + \theta R & -R + q_5 \end{pmatrix} \\
&= -(R + q_{1J}) \left[(R + q_3)(R - q_5) + q_4 (q_6 + \theta R) \right] \\
&\quad - q_2 \left(-R q_3 + q_3 q_5 - q_4 q_6 \right) \\
&= -(R + q_{1J}) \left[R^2 - R m_1 - m_2 \right] + q_2 (R q_3 - m_2) \\
&= -R^3 + (m_1 - q_{1J}) R^2 + (m_2 + m_1 q_{1J} + q_2 q_3) R + m_2 (q_{1J} - q_2),
\end{aligned}$$

where

$$\begin{aligned}
m_1 &\equiv q_5 - q_3 - \theta q_4, \\
m_2 &\equiv q_3 q_5 - q_4 q_6 \\
&= (g + \delta_H) \left(r^{bg} + \delta_H \right) \Gamma_3 \zeta (\Gamma_2 - \eta) > 0,
\end{aligned}$$

and the last line uses the fact that $\Gamma_3 < 0$ and $\Gamma_2 - \eta < 0$.

Write the cubic as

$$0 = \Psi_J(R) \equiv -R^3 + A_{2J} R^2 + A_{1J} R + A_{0J}, \qquad J = H, L,$$

where

$$\begin{aligned}
A_{2J} &\equiv m_1 - q_{1J}, \\
A_{1J} &\equiv m_2 + m_1 q_{1J} + q_2 q_3, \\
A_{0J} &\equiv m_2 (q_{1J} - q_2), \qquad J = H, L.
\end{aligned}$$

Since $a_H^{bg} > 1/2 > a_L^{bg}$, it follows that $q_{1H} > q_2 > q_{1L}$. Hence

$$\Psi_H(0) = A_{0H} > 0, \qquad \Psi_L(0) = A_{0L} < 0,$$

so Ψ_H has at least one positive real root, and Ψ_L has at least one negative real root.

The other roots of Ψ_H are real and both are negative if and only if Ψ_H has real inflection points, the smaller one is negative—call it $I_H < 0$, and $\Psi_H(I_H) < 0$. Similarly, the other roots of Ψ_L are real and both are positive, if and only if Ψ_L has real inflection points, the larger one is positive—call it $I_L > 0$, and $\Psi_L(I_L) > 0$. The inflection points are solutions of the quadratic $\Psi_J'(I) = 0$, so they are real if and only if $\Psi_J'(0) = A_{1J} > 0$. Write this condition as

$$0 < m_2 + (m_1 + q_3) q_2 + m_1 (q_{1J} - q_2). \tag{27}$$

To construct examples where Ψ_J has complex roots, consider cases where the inequality in (27) fails. As noted above, $m_2 > 0$. The term $(q_{1J} - q_2)$ is small in absolute value if a_J^{bg} is close to 1/2. As shown in Fig. 4b, this occurs if $1/B$ is close to the lowest value for which a BGP exists. For the middle term note that $q_2 > 0$ and

$$
\begin{aligned}
m_1 + q_3 &= q_5 - \theta q_4 \\
&= (g + \delta_H)\,\Gamma_3\,(\Pi_H \Gamma_2 - \theta\eta) \\
&= -\,(g + \delta_H)\,\Gamma_3\,[\theta\eta - \Pi_H\eta + \chi + \Pi_H - \eta] \\
&= -\,(g + \delta_H)\,\Gamma_3\left[\eta\left(\theta - 1 + \frac{\omega}{1-\sigma}\right) + \Pi_H\,(1-\eta)\right].
\end{aligned}
$$

Since $\Gamma_3 < 0$, this term has the sign of the expression in square brackets. The term $\Pi_H\,(1-\eta)$ is positive, but for η close to one it is small. The term $\omega/\,(1-\sigma)$ is also positive, but for ω close to zero it is also small. The term $\theta - 1$ is negative if θ is close to zero. Thus, complex roots can occur if B is large, η is close to unity, and ω, θ are close to zero.

For example, with the preference and technology parameters

$$\theta = 0.10,\ \eta = 0.95,\ \zeta = 0.05,\ \omega = 0.05,\ \beta = 0.45,\ a_F = 0.9179,$$

and the others at the baseline values, and the policy parameters

$$\sigma = 0.9026,\ B = 1.0,$$

the steady states are $a_H^{bg} = 0.6684$ and $a_L^{bg} = 0.3316$, and the system has complex roots—with negative real parts—at both steady states. For nearby policy parameters, the complex roots can have positive real parts at the low steady state, and the roots at the high steady state can be real while those at the low steady state remain complex. Notice that ω close to zero implies that an individual's wage depends very little on his own human capital, while $1 - \omega - \beta = 0.50$ implies that average human capital is quite important for productivity. Hence the optimal steady state subsidy is quite high—here it is $\sigma_F = 0.9163$, and the example uses a subsidy almost that high. In addition, θ close to zero implies a very high elasticity of intertemporal substitution. These parameter values are implausible.

Appendix 3: Computing transition paths to the SS

The following two-step procedure is used to compute transition to the stagnation SS. For the first step, let $A_0 = A^{st}$ and conjecture that $\dot{A} = 0$. Then the transition is described by the equations for $\dot{H}$, $\dot{\Lambda}_H$, which do not involve W. Linearize (3) and (22) around $\left(H^{st}, \Lambda^{st}\right)$, using (21) for v. The resulting pair of equations has roots that are real and of opposite sign. Hence for any H_0 sufficiently close to H^{st}, there exists a unique Λ_{H0} near Λ_H^{st} with the property that for the initial conditions (H_0, Λ_{H0}) and $A_0 = A^{st}$, the linearized system converges to the steady state. This solution constitutes a competitive equilibrium provided that $\dot{A} = 0$ when (2) is used. That holds provided that $H_0/\,W_0$ is small enough. For the second step, choose any pair (H, Λ_H) generated by the first step, and a (large) value W. Run the system of ODEs in (1)–(3) and (22) backward from the 'initial' condition $\left(W, A^{st}, H, \Lambda_H\right)$.

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