Exotic options and Lévy processes

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1   Introduction

A number of empirical studies have shown that the usual geometric Brownian motion model used for the pricing of derivative securities is not appropriate in many markets. As these products are becoming more and more popular, it is no longer acceptable for a financial institution to use the geometric Brownian motion regardless of its drawbacks. In the equities market, the smile phenomenon stands out amongst the main concerns: as European options have become liquid and can be treated as primary assets, it is important that a model matches their prices if this is to be used to price (and hedge) exotic options. Several kinds of alternative models have been and are still being developed, among which models with stochastic volatility (e.g. [37], [38]) or Lévy processes: Barndorff-Nielsen [5] introduced the NIG model, Madan and Seneta [20] proposed the VG model, extended in [32]. For an argument in favor of using Lévy processes in financial modelling, see [44]. Some models have also been investigated, which mix stochastic volatility and jumps ([9], [25], [29], [59]). Some of these models and their performance are reviews in other chapters of the present handbook [].

Among exotic options, barrier and lookback options have the simplest structure and were introduced a long time ago; they are today quite popular among investors since they reduce the unwanted part of the risk carried by vanilla options. Not only traded on their own, they are commonly involved in more elaborate structured products.
In the framework of the geometric Brownian motion model, the problem has been addressed and solved many times. Kunitomo and Ikeda [42] derive prices for barrier options with general boundaries (see also [3]); Geman-Yor [33] and Sbuelz [58] use the Laplace transform method to price double barrier options; Pelsser [52] derives the same formulas using path integration.

Here, we deal with the pricing of barrier and lookback options when the underlying asset price is modelled as e^{X_t} for a Lévy process X. By making use of purely probabilistic techniques, we obtain results that hold in the most general case, which embeds many popular models, such as the geometric Brownian motion (!), jump-diffusion, the Normal Inverse Gaussian model and the Variance-Gamma distribution.
Thanks to the Pecherskii-Rogozin identity, we obtain a formula for the Laplace

transform of option prices in terms of the Wiener-Hopf factors or the Laplace exponent of the ladder process associated with X.

This work is organized as follows. Lévy processes and their main mathematical properties, as well as fluctuation theory are introduced in Section 2. We also give some definitions and notations to be used in the sequel. In Section 3 we review some uses of Lévy processes in finance, before describing barrier and lookback options, and showing how their prices can be derived through Laplace inversion. Pricing in particular cases is also studied, but the most general method gives some insight about hedging. Some examples are studied in Section 4, where some formulas –as explicit as possible– are obtained for the Laplace transform of the option price. Section 5 concludes. Subsequently, a numerical example illustrates our technique in Appendix A, while Appendix B points to possible extensions and future research.

2 Lévy processes

In this section, we introduce Lévy processes and review their main mathematical properties. We put some emphasis on fluctuation theory, as this is the leading "technology" which enables us to solve our valuation problem.

2.1 Definition and main properties

In this section, we define Lévy processes and give a short review of their main properties. More details can be found in the books by Bertoin [11] and Sato [56] (see also [12], [57] for a short introduction) .

We assume a probability space $(\Omega, \mathcal{F}, P)$ is given.

Definition 2.1 *A real-valued process $X = (X_t, t \geq 0)$ is called a Lévy process if it satisfies the following conditions*

- *for all $s \geq 0$, the shifted process $(X_{t+s} - X_s, t \geq 0)$ is independent of $(X_u, u \leq s)$;*

- *for all $0 \leq s \leq t$, the distribution of $X_t - X_s$ coincides with the distribution of X_{t-s};*

- *X has a.s. right-continuous paths.*

If X is a Lévy process, the distribution of X_1 is infinitely divisible because of the identity

$$
\begin{aligned}
X_1 \quad &= \quad X_{1/n} + (X_{2/n} - X_{1/n}) + \cdots + (X_1 - X_{1-1/n}) \\
&\overset{\text{law}}{=} \quad X_{1/n}^{(1)} + X_{1/n}^{(2)} + \cdots + X_{1/n}^{(n)}
\end{aligned}
$$

where the $X_{1/n}^{(i)}$ are mutually independent variables, all distributed like $X_{1/n}$. Conversely, to any infinitely divisible distribution μ on $\mathbb{R}$ one can associate a

Lévy process X such that the law of X_1 is μ. Infinite divisibility of the law of X_1 is equivalent to the Lévy-Khintchine representation of its characteristic function:

$$E[e^{iuX_1}] = e^{-\phi(u)}$$

where

$$(1) \qquad \phi(u) = iau + \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}-\{0\}}(1 - e^{iux} + iux1_{|x|<1})\nu(dx).$$

In the above equation, a is a real number (the *drift*), $\sigma \geq 0$ (the *diffusion coefficient*) and ν is a σ-finite measure on $\mathbb{R} - \{0\}$ (the *Lévy measure*) which satisfies the condition

$$(2) \qquad \int (1 \wedge x^2)\nu(dx) < \infty$$

We then have for each t

$$(3) \qquad E[e^{iuX_t}] = e^{-t\phi(u)}.$$

ϕ is called the characteristic or Lévy exponent of X.

The Lévy-Khintchine representation of the characteristic exponent ϕ has a corresponding path interpretation: X can be decomposed as

$$(4) \qquad X_t = at + \sigma B_t + \sum_{s \leq t}\Delta X_s 1_{|\Delta X_s|>1} + \lim_{\epsilon \downarrow 0} Z_t^{(\epsilon)}$$

where

- B is a standard Brownian motion,

- the sum makes sense because X has right-continuous paths hence finitely many jumps of absolute size > 1 in any finite time interval,

- for each $\epsilon > 0$, $Z^{(\epsilon)}$ is the martingale defined by

$$Z_t^{(\epsilon)} = \sum_{s \leq t}\Delta X_s 1_{\epsilon<|\Delta X_s|\leq 1} - t\int_{\epsilon<|x|\leq 1} x\nu(dx);$$

one can show that these martingales converge in L^2, uniformly on any bounded time interval, as ϵ goes to 0; the limit is a pure jump martingale.

Note that the 3 processes above are mutually independent Lévy processes. The decomposition (4) is called the Lévy-Ito decomposition of the paths of X. More details are given in [56, §§19-20].

From now on, we denote by $(\mathcal{F}_t, t \geq 0)$ the filtration generated by X. Let us mention two well-known important properties of Lévy processes. The first one is the strong Markov property in the filtration $\mathcal{F}_t$. More precisely, for any $\mathcal{F}_t$-finite stopping time T, the shifted process $(X_{T+t} - X_T, t \geq 0)$ is independent of $\mathcal{F}_T$ and has the same law as $(X_t - X_0, t \geq 0)$. The second one is that X is a special semimartingale (see [48, p. 310]). The semimartingale property can be read from the Lévy-Ito decomposition; it is a special semimartingale because the compensator of the jumps is deterministic (cf. [39]).

2.2 Fluctuation theory

We now turn to a branch of the theory of Lévy processes, named fluctuation theory. Its aim is to study the joint behaviour of a process (here, our Lévy process) and its maximum. Fluctuations of random walks were first studied, for which Spitzer obtained many important results. Thanks to the independence of increments, most of these results can be transposed to Lévy processes. However, these results can be rediscovered by using excursion theory, an approach we favor because it involves directly the paths of X. We follow closely Bertoin [11, Chap. VI] and Greenwood-Pitman [36]. For an account on fluctuation theory for continuous time processes, see Bingham [13].

Let

$$(5) \qquad M_t = \sup_{s \le t} X_s$$

the running maximum of the Lévy process X. The key point in the following is that the reflected process $M - X$ possesses the strong Markov property with respect to the filtration $(\mathcal{F}_t)$. Let L be a local time process at 0 of the reflected process and denote by τ its right-continuous inverse:

$$(6) \qquad \tau_t = \inf\{u > 0, L_u > t\}.$$

Put

$$(7) \qquad H(t) = M_{\tau(t)} \quad \text{if } \tau(t) < \infty, \quad H(t) = \infty \quad \text{otherwise.}$$

The process (τ, H) is called the ladder process, it is a two-dimensional Lévy process, each component being a subordinator. Roughly speaking, H is the process of the successive values of the supremum of X, when the intervals of time where M is constant have been discarded; this is revealed by the use of the inverse local time as a time scale instead of the original calendar time.

Let κ be the Laplace exponent of the ladder process:

$$e^{-t\kappa(\alpha, \beta)} = E\big[e^{-\alpha\tau(t) - \beta H(t)}\big].$$

One of the main goals of fluctuation theory is to compute the function κ. Knowledge of this function will be needed in order to apply the main result of this section, the Pecherskii-Rogozin identity (Theorem 2.1).

We first proceed to give an expression of κ in terms of the 1-dimensional distributions of X.

Let θ be a random variable independent of X, exponentially distributed with parameter $q > 0$ and denote $G_\theta = \sup\{t < \theta : X_t = M_t\}$. It is easy to see that $M_\theta = M_{G_\theta-} = X_{G_\theta-}$. Since the excursion process away from 0 associated with $M - X$ is a Poisson point process, we have that $(X_t, t < G_\theta)$ and $(X_{G_\theta+t} - X_{G_\theta-}, t \le \theta - G_\theta)$ are independent. In particular, the pairs of variables (G_θ, M_θ) and $(\theta - G_\theta, X_\theta - M_\theta)$ are independent. The following result will be used in section 4 for a number of models.

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Proposition 2.1 *There exists a constant $k > 0$ such that:*

$$(8) \qquad \kappa(\alpha, \beta) = k \exp\left(\int_0^\infty dt \int_0^\infty t^{-1}(e^{-t} - e^{-\alpha t - \beta x})P(X_t \in dx)\right)$$

Proof. We have a decomposition of (θ, X_θ) as the sum of two independent, infinitely divisible random variables (G_θ, M_θ) and $(\theta - G_\theta, X_\theta - M_\theta)$. Denoting by μ, μ^+, μ^- the respective Lévy measures of these variables, we then have $\mu = \mu^+ + \mu^-$, with μ^+ (resp. μ^-) having support in $[0, \infty) \times [0, \infty)$ (resp. $[0, \infty) \times (-\infty, 0]$). The Lévy measure of (θ, X_θ) is given by $t^{-1}e^{-qt}P(X_t \in dx)dt$ (see [11, Lemma VI.7]) and can be decomposed as

$$t^{-1}e^{-qt}P(X_t \in dx)dt1_{x>0} + t^{-1}e^{-qt}P(X_t \in dx)dt1_{x<0};$$

from which we conclude that $\mu^+ = t^{-1}e^{-qt}P(X_t \in dx)dt1_{x>0}$.
On the other hand, one can show using the theory of excursions, that the Laplace transform of (G_θ, M_θ) is given by

$$(9) \qquad E[e^{-\alpha G_\theta - \beta M_\theta}] = \frac{\kappa(q, 0)}{\kappa(\alpha + q, \beta)}.$$

(this can be proved using a decomposition over excursion intervals as done in the proof of Theorem 2.1 below). Putting pieces together, we have

$$\frac{\kappa(q, 0)}{\kappa(\alpha + q, \beta)} = \exp\left(- \int (1 - e^{-\alpha t - \beta x})\mu^+(dt, dx)\right)$$

because there is no drift term in the distribution of (G_θ, M_θ). So

$$\kappa(\alpha + q, \beta) = \kappa(q, 0) \exp\left(\int_0^\infty dt \int_0^\infty P(X_t \in dx)t^{-1}e^{-qt}(1 - e^{-\alpha t - \beta x})\right)$$

for all $\alpha, \beta, q > 0$. Setting $q = 1$, we get the result for $\alpha > 1$ and $\beta > 0$. A standard argument of analytic continuation entails that the proposition is true for all $\alpha > 0$. $\qquad \square$

Remark 2.1 *The constant $k = \kappa(1, 0)$ is arbitrary as it depends only on the normalization of the local time process L. Upon multiplying this process by some positive constant, we can choose $k = 1$. However, we keep this kind of constants in our general formulas; they will automatically (and consistently) be discarded in the pricing formulas.*

We now turn to prove the Pecherskii-Rogozin identity, which expresses the double Laplace transform of the joint distribution of hitting times and the value of the process at such times in terms of the function κ. This will be our main tool for pricing barrier and lookback options in a general setup. This identity

has been known for quite a long time and was first proved in [51] using Wiener-Hopf techniques of analysis. The proof we give here is based on the theory of excursions and avoids the original analytic arguments. See also [56, §49] for an alternative proof.

From now on, we suppose that 0 is regular for $(0, \infty)$ relatively to X, i.e. $\inf\{t > 0 : X_t > 0\} = 0$ a.s., and also that 0 is instantaneous for the reflected process $M - X$, i.e. if $\inf\{t > 0 : M_t - X_t \neq 0\} = 0$ a.s. If 0 is irregular, or is not instantaneous, the set $\{t : X_t = M_t\}$ is discrete, and elementary arguments based on the strong Markov property at the successive passage times at 0 of the process $M - X$ suffice to prove Theorem 2.1. For more details, see [11, Chap. IV].

Theorem 2.1 (Pecherskii-Rogozin) *For $x > 0$, define $T(x) = \inf\{t > 0 : X_t > x\}$ the first passage time above x and $K(x) = X_{T(x)} - x$ the so-called overshoot. For every $\alpha, \beta, q > 0$, the following formula holds:*

$$(10) \qquad \int_0^\infty e^{-qx} E[e^{-\alpha T(x) - \beta K(x)}] dx = \frac{\kappa(\alpha, q) - \kappa(\alpha, \beta)}{(q - \beta)\kappa(\alpha, q)}$$

Before we prove the Pecherskii-Rogozin identity, we need the following results:

Proposition 2.2 *For $x > 0$, let $\eta_x := \inf\{t : H(t) > x\}$.*

1. *The process $Z_x = H(\eta_x) - x$ is a Markov process in the filtration $(\mathcal{F}_{\eta_x}, x > 0)$.*

2. *For all $x > 0$, it holds a.s. that $T(x) = \tau(\eta_x)$ and $K(x) = Z_x$.*

Remark 2.2 *The identity in the above proposition is easy to understand from the intuitive interpretation of the process (τ, H) given above. Indeed, the time $T(x)$ is an increase time point for M. The equality $T(x) = \tau(\eta_x)$ is just the change of time-scale between M and H; the overshoot is expressed, in the new time-scale, as*
$$K(x) = M_{T(x)} - x = H_{\eta_x} - x.$$

Proof .

1. This follows immediately from the general theory of time changes for Markov processes.

2. We first show that $T(x)$ is in the range of τ. Since $T(x)$ is a zero of $M - X$ it is enough to show that it is not the left end-point of an excursion interval –it will then be a right end-point. We will therefore have $T(x) = \tau(l(x))$ and also $l(x) = L(T(x))$.
   Suppose on the contrary that $T(x)$ is the left end-point of an excursion interval $(\tau(a-), \tau(a))$ (so $(T(x) = \tau(a-))$. We then have $M > X$ on an

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open interval $(T(x), \tau(a))$. On the other hand, $T(x)$ is a stopping time and the strong Markov property yields

$$(M_{T(x)+t} - X_{T(x)+t}, t \geq 0) \stackrel{d}{=} (M_t - X_t, t \geq 0).$$

This is in contradiction with the fact that 0 is regular for $M - X$, relatively to $[0, \infty)$.

Let us now show that $l(x) = \eta_x$, which will end the proof. We distinguish between two cases:

- $H(l(x)) > x$. Then $H(l(x)) = M_{\tau(l(x))} = M_{T(x)} = X_{T(x)} > x$. On the other hand, $X_t \leq x$ for all $t < T(x) = \tau(l(x))$ and for $t < l(x)$ we have $\tau(t) < \tau(l(x))$. If $H(t) > x$, then $T(x) \leq \tau(t)$, which does not hold, so that $H(t) \leq x$ for $t < l(x)$, meaning that $l(x) = \eta_x$.

- $H(l(x)) = x$. As in the previous case, we have $H(t) \leq x$ for $t < l(x)$. On the other hand, the Markov property implies that X visits $[0, \infty)$ immediately after $T(x)$ a.s. So for $t > l(x)$, we have $H(t) = M_{\tau(t)} > M_{\tau(l(x))} = x$, i.e. $l(x) = \eta_x$.

$\qquad \square$

Lemma 2.1 *For all $u \geq 0$, $\eta_{M_{\tau(u)}} = \eta_{M_{\tau(u-)}} = u$*

Proof . We have:

$$\eta_{M_{\tau(u)}} = \inf\{t : H(t) > H(u)\} = u$$

and

$$\eta_{M_{\tau(u-)}} = \inf\{t : H(t) > H(u-)\} \geq u$$

because H is increasing. Also, we have $\eta_{M_{\tau(u-)}} \leq \eta_{M_{\tau(u)}}$ so the proof is complete.
$\square$

Lemma 2.2 *η_x is continuous, and ν^H, the Lévy measure of H satisfies*

$$\int_0^\infty \nu^H(dx) = +\infty.$$

Proof . We have that $H(s) < H(t)$ for any $s < t$; it follows that η is continuous. Now, the jump part of H is not a compound Poisson process because of the hypothesis that 0 is a regular point for $(0, \infty)$ relatively to $M - X$. Hence its Lévy measure cannot be finite. $\qquad \square$

Lemma 2.3 *The following identity between measures holds a.s:*

$$dx \mathbf{1}_{H(\eta_x)=x} = d^H d\eta_x$$

where d^H is the drift of the Lévy process H.

Proof . For $a > 0$, denote by $l_1(a)$ the size of the first jump of H which is greater than a. Fix $c > 0$, then for $a < c$, since the jump part of H is a Poisson point process with intensity measure ν^H:

$$P[l_1(a) > c] = \frac{\nu^H([c, \infty))}{\nu^H([a, \infty))} \to 0, \quad a \to 0$$

so that 0 is a regular point for the process Z introduced in Proposition 2.2. On the other hand, for $a > c$,

$$P[l_1(c) > a] = \frac{\nu^H([a, \infty))}{\nu^H([c, \infty))} \to 0, \quad a \to \infty$$

so that 0 is also recurrent for Z. It is easily seen that the sets $\{x : Z_x = 0\}$ and $\{x : H(\eta_x) = x\}$ coincide, and that η increases only on the set $\{x, \exists t : H(t) = H(t-) = x\} = \{x : H(\eta_x) = x\}$. So η is a local time process for Z (see [11, Chap. IV]). Hence, ([11, Prop. IV.6])

$$\int_0^x 1_{(Z_y=0)} dy = d^H \eta_x$$

where d^H is the drift of H. This concludes the proof. $\square$

There is a more intuitive argument for the above result. Indeed, as a subordinator, H has the Lévy-Ito representation

$$H_t = d^H t + \int_0^t \int h \, (\mu - \nu^H)(ds, dh)$$

where ν^H is the Lévy measure of H and μ is a random Poisson measure with intensity ν^H. It follows that η has locally a representation

$$\eta_x = \frac{1}{d^H} x 1_{x \in A}$$

where $A = \{x : H(\eta_x) = x\}$ and is constant on the set corresponding to jumps of H. We conclude that $d^H d\eta_x = 1_{H(\eta_x)=x} dx$.

Proof of Theorem 2.1. First we split the quantity of interest over the excursion intervals of the process Z in the above lemma:

$$\int_0^\infty e^{-qx} E[e^{-\alpha T(x) - \beta K(x)}] dx = I + II$$

where

$$I := E\left[\int_0^\infty e^{-qx} e^{-\alpha \tau(\eta_x) - \beta(H(\eta_x)-x)} 1_{H(\eta_x)>x} dx\right]$$

and

$$II := E\left[\int_0^\infty e^{-qx} e^{-\alpha \tau(\eta_x) - \beta(H(\eta_x)-x)} 1_{H(\eta_x)=x} dx\right].$$

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The first term I may be split over the excursion intervals according to

$$I = E\left[\sum_{u>0}\int_{H(u-)}^{H(u)} e^{-qx-\alpha\tau(\eta(x))-\beta(H(\eta_x)-x)}dx\right]$$

Let us perform the change of variables $x = H(u) - (h - H(u-))$, so that

$$\begin{aligned}
I &=& E\left[\sum_{u>0}\int_{H(u-)}^{H(u)} e^{-q[H(u)-h+H(u-)]-\alpha\tau(\eta_{H(u)-h+H(u-)})-\beta[H(\eta_{H(u)-h+H(u-)})-H(u)+h-H(u-)]}dh\right]\\
&=& E\left[\sum_{u>0}\int_{H(u-)}^{H(u)} e^{(q-\beta)(h-H(u-))-\alpha\tau(u)-qH(u)}dh\right]
\end{aligned}$$

since according to Lemma 2.1, we have $\eta_{H(u)} = \eta_{H(u-)} = u = \eta_{H(u)-(h-H(u-))}$ for all $h \in (H(u-), H(u))$. So

$$I = E\left[\sum_{u>0}e^{(q-\beta)H(u-)-\alpha\tau(u-)-qH(u-)-\alpha(\tau(u)-\tau(u-))-q(H(u)-H(u-))}\int_0^{H(u)-H(u-)} e^{(q-\beta)h}dh\right]$$

Let $\Phi(t, h) = e^{-\alpha t - qh}$ and apply the compensation formula ([11, Chap. 0])to get

$$\begin{aligned}
I &=& E\left[\int_0^\infty du\Phi(\tau(u), H(u))\int\nu(dt, dh)e^{-(\alpha t+qh)}\int_0^h e^{(q-\beta)v}dv\right]\\
&=& \frac{1}{(q-\beta)\kappa(\alpha, q)}\int\nu(dt, dh)e^{-\alpha t}(e^{-\beta h} - e^{-qh})
\end{aligned}$$

where $\nu(dt, dh)$ is the Lévy measure of the bivariate Lévy process (τ, H). The Lévy-Khintchine representation for κ reads

$$\kappa(\alpha, \beta) = d^\tau\alpha + d^H\beta + \int(1 - e^{-\alpha t-\beta h})\nu(dt, dh)$$

wherefrom we deduce that

$$\int\nu(dt, dh)e^{-\alpha t}(e^{-\beta h} - e^{-qh}) = \kappa(\alpha, q) - \kappa(\alpha, \beta) + (\beta - q)d^H$$

and we finally obtain

$$(11)\qquad\qquad I = \frac{\kappa(\alpha, q) - \kappa(\alpha, \beta)}{(q-\beta)\kappa(\alpha, q)} - \frac{d^H}{\kappa(\alpha, \beta)}.$$

Let us now consider the second term. According to Lemma 2.3:

$$\begin{aligned}
II &=& E\left[\int_0^\infty e^{-qx-\alpha\tau(\eta_x)-\beta(H(\eta_x)-x)}1_{H(\eta_x)=x}dx\right]\\
&=& E\left[\int_0^\infty e^{-qx-\alpha\tau(\eta_x)}d^H d\eta_x\right].
\end{aligned}$$

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Since η is continuous, we can perform the change of variables $u = \eta_x$ in the integral; this yields

$$
\begin{aligned}
II &= d^H E\left[\int_0^\infty e^{-qH(u)-\alpha\tau(u)}du\right] \\
&= \frac{d^H}{\kappa(\alpha,q)}.
\end{aligned}
$$

(12)

Combination of Equations (11) and (12) entails the Pecherskii-Rogozin identity. □

Remark 2.3 *When the process X has no positive jumps, we have $X_{T(x)} = x$ a.s. This case is considered in a simpler manner in the next section.*

Denote by $\hat{X}$ the dual Lévy process of X, namely $\hat{X}_t = -X_t$ and by $\hat{\kappa}$ the Laplace exponent of the ladder process associated with $\hat{X}$. According to Proposition 2.1, we have

$$
\begin{aligned}
\hat{\kappa}(\alpha,\beta) &= \hat{k}\exp\left(\int_0^\infty dt\int_0^\infty t^{-1}(e^{-t}-e^{-\alpha t-\beta x})P(-X_t\in dx)\right) \\
&= \hat{k}\exp\left(\int_0^\infty dt\int_{-\infty}^0 t^{-1}(e^{-t}-e^{-\alpha t-\beta x})P(X_t\in dx)\right).
\end{aligned}
$$

In particular, we have

$$
\begin{aligned}
\kappa(\alpha,0) &= k\exp\left(\int_0^\infty t^{-1}(e^{-t}-e^{-\alpha t})P(X_t\geq 0)dt\right) \\
\hat{\kappa}(\alpha,0) &= \hat{k}\exp\left(\int_0^\infty t^{-1}(e^{-t}-e^{-\alpha t})P(X_t\leq 0)dt\right)
\end{aligned}
$$

so that

$$
\begin{aligned}
\kappa(\alpha,0)\hat{\kappa}(\alpha,0) &= k\hat{k}\exp\left(\int_0^\infty t^{-1}(e^{-t}-e^{-\alpha t})dt\right) \\
&= k\hat{k}\alpha
\end{aligned}
$$

(13)

where the last equality follows from using the Frullani integral: $a\ln(1+\lambda/b) = \int_0^\infty(1-e^{-\lambda t})at^{-1}e^{-bt}dt$. On the other hand, it is an easy consequence of the independence of M_θ and $X_\theta - M_\theta$ that the following holds:

(14)
$$
\frac{q}{q+\phi} = \phi_q^+\phi_q^-
$$

where ϕ_q^+ (resp. ϕ_q^-) is the characteristic function of M_θ (resp. $X_\theta - M_\theta$). This identity is known as the Wiener-Hopf factorization. From equation (9) we deduce that

(15$^+$)
$$
\phi_q^+(u) = \frac{\kappa(q,0)}{\kappa(q,-iu)}
$$

and using duality, we have similarly

$$(15^-) \qquad\qquad \phi_q^-(u) = \frac{\hat{\kappa}(q,0)}{\hat{\kappa}(q,iu)}.$$

So combining equations (13), (14), (15^+) and (15^-), $\hat{\kappa}$ can be deduced from κ by the formula

$$(16) \qquad\qquad \kappa(q,-iu)\hat{\kappa}(q,iu) = k\hat{k}(q + \phi(u)).$$

2.3 Lévy processes with no positive jumps

We now turn to the case when the Lévy process X has no positive jumps (X is then called spectrally negative); in this case, the support of the Lévy measure is contained in $(-\infty, 0)$. Note that by considering the dual process, this study applies to the case when X has no negative jumps. The results we present here already appear in e.g. [62], where they are obtained by a fine analysis of the Laplace exponent ψ; we use quite different techniques that rely on martingale theory; see also [11].

Since there are no positive jumps, we can consider the Laplace exponent ψ, defined by

$$(17) \qquad\qquad E[e^{\lambda X_t}] = e^{t\psi(\lambda)}, \qquad \lambda > 0;$$

ψ is well-defined because X has no positive jumps and the Laplace transform of a Gaussian variable is finite. This formula holds indeed for all complex λ such that $\Re(\lambda) > 0$.

The equation $\psi(\lambda) = 0$ has at least one solution; denote the largest one by $\Phi(0)$. Then ψ is a bijection when restricted to $[\Phi(0), \infty)$ and we denote by Φ the inverse bijection. Standard martingale arguments will yield the Laplace transform of the first passage times $T(x)$.

Indeed, since $\psi(\lambda) < \infty$, it is clear that $e^{\lambda X_t - t\psi(\lambda)}$ is a martingale; because $T(x)$ is a stopping time, the optional sampling theorem gives

$$(18) \qquad\qquad E[e^{\lambda X_{T(x)} - T(x)\psi(\lambda)}1_{T(x)<\infty}] = 1.$$

Now because X has no positive jumps, the equality $X_{T(x)} = x$ holds a.s. on $\{T(x) < \infty\}$, so that the preceding equation can be written

$$(19) \qquad\qquad E[e^{-T(x)\psi(\lambda)}1_{T(x)<\infty}] = e^{-\lambda x}$$

or, considering the inverse bijection:

$$(20) \qquad\qquad E[e^{-\lambda T(x)}1_{T(x)<\infty}] = e^{-x\Phi(\lambda)}.$$

Note that the set $\{T(x) = \infty\}$ is not necessarily P-negligible: it has positive probability if and only if $\Phi(0) > 0$, and then the law of $T(x)$ has an atom at $+\infty$: $P[T(x) = +\infty] = 1 - e^{-x\Phi(0)}$.

In the case of no positive jumps, we have the following nice property of the running maximum.

Proposition 2.3 *Let θ be an exponential variable with parameter q, independent of X. Then M_θ has an exponential distribution with parameter $\Phi(q)$.*

Proof. For $x > 0$, we have the following chain of equalities

$$
\begin{aligned}
P(M_\theta \geq x) &= P(T_x \leq \theta) \\
&= \int_0^\infty q e^{-qt} P(T_x \leq t) dt \\
&= E\left[\int_{T_x}^\infty q e^{-qt} dt\right] \qquad \text{by Fubini's theorem} \\
&= E[e^{-qT_x} 1_{(T_x < \infty)}] \\
&= e^{-x\Phi(q)}
\end{aligned}
$$

as required. $\qquad\square$

We end this section by the following remark. The absence of positive jumps ensures that the ladder height process is given by $H_t = M_{\tau(t)} = t$ for $\tau(t) < \infty$. The bivariate Laplace exponent κ is now especially simple.

Theorem 2.2 *The bivariate Laplace exponent κ of the ladder process is given by*

$$(21) \qquad\qquad \kappa(\alpha, \beta) = \Phi(\alpha) + \beta.$$

The bivariate Laplace exponent of the dual ladder process is given by

$$(22) \qquad\qquad \hat{\kappa}(\alpha, \beta) = k\hat{k}\frac{\alpha - \psi(\beta)}{\Phi(\alpha) - \beta}.$$

Proof. Because X has no positive jumps, its supremum functional M is continuous and additive for $M - X$. Moreover it is easy to see that M increases exactly on the set $\{t : X_t = M_t\}$. Therefore M can be used as a local time for the reflected process $M - X$ (see [11, Th. VII.1]). It follows immediately that the ladder height process is simply $H(t) = t$. The inverse local time is then seen to coincide with the passage time process $T(x)$, and we have shown above that

$$E[e^{-\alpha T(x)}] = e^{-x\Phi(\alpha)}.$$

It follows that $\kappa(\alpha, \beta) = \beta + \Phi(\alpha)$. Now recall identity (16) to get

$$
\begin{aligned}
\hat{\kappa}(\alpha, \beta) &= k\hat{k}\frac{\alpha + \phi(-i\beta)}{\kappa(\alpha, -\beta)} \\
&= k\hat{k}\frac{\alpha - \psi(\beta)}{\Phi(\alpha) - \beta}
\end{aligned}
$$

$\qquad\square$

Hence, to deal with the case when the Lévy process X has no negative jumps, we may simply consider that it is the dual process of a Lévy process $\hat{X}$ with no positive jumps.

2.4 Esscher transforms (or: Exponential tilting)

In this section we introduce a family of measure transformations that preserve the Lévy property, i.e. X is still a Lévy process under the new measure. Fix $\lambda > 0$, and assume that for some (hence all) $t > 0$:

$$E[e^{\lambda X_t}] < \infty.$$

Then $e^{\lambda X_t - t\psi(\lambda)}$ is a P-martingale, where $E[e^{\lambda X_t}] = e^{t\psi(\lambda)}$, and we can define a probability measure $P^{(\lambda)}$ by the formula

$$(23) \qquad P^{(\lambda)}|_{\mathcal{F}_t} = \frac{e^{\lambda X_t}}{E[e^{\lambda X_t}]} \cdot P|_{\mathcal{F}_t}$$

For a general view on Esscher-type transforms of probability measure, see e.g. [40] or [16]. The interesting point in the Esscher transform is the following

Theorem 2.3 *Let $\lambda > 0$ such that $E[e^{\lambda X_T}] < \infty$ and define the probability measure $P^{(\lambda)}$ by the formula (23). Then X is a Lévy process under $P^{(\lambda)}$.*

Furthermore if the characteristic exponent of X under P has a Lévy-Khintchine representation (1):

$$\phi(u) = iau + \frac{\sigma^2}{2}u^2 + \int (1 - e^{iux} - iux1_{|x|<1})\nu(dx),$$

then the characteristic exponent of X under $P^{(\lambda)}$ is given by

$$(24) \qquad \phi^{(\lambda)}(u) = \phi(u - i\lambda) - \phi(-i\lambda),$$

and has a Lévy-Khintchine representation

$$(25) \qquad \phi^{(\lambda)}(u) = ia^{(\lambda)}u + \frac{1}{2}\sigma^2 u^2 + \int (1 - e^{iux} + iux1_{|x|<1})\nu^{(\lambda)}(dx)$$

where

$$(26) \qquad a^{(\lambda)} = a + \sigma + \int (e^{\lambda x} - 1)\nu(dx)$$

$$(27) \qquad \nu^{(\lambda)}(dx) = e^{\lambda x}\nu(dx)$$

Theorem 2.3 is a special case of Theorems 33.1 and 33.2 in [56] which describe, in general, the Radon-Nikodym density –when it exists– between two Lévy processes. The Esscher transform occurs when the density between the two Lévy measures is of the form $e^{\lambda x}$.

By Theorem 2.3, we will be able to study the functionals of X in which we are interested in this paper under the measure $P^{(\lambda)}$. In particular we will have access to the distribution of $T(x)$ under $P^{(\lambda)}$. We will then be able to use the technique of a change of numéraire which will be quite useful when computing the price of options, see Section 3.6.2.

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3 Pricing of Barrier and Lookback options

In this section, we present barrier and lookback options, and explain how Theorem 2.1 can be used to derive their price in a fairly general setting. We begin by introducing the framework in which this will be done.

3.1 The model

We consider a financial market the uncertainty of which is described by a probability space $(\Omega, \mathcal{F}, P)$ and that consists of one riskless asset with a constant (risk-free) rate of return r and a risky asset whose price process S_t satisfies

$$(28) \qquad\qquad S_t = S_0 e^{X_t}$$

where X is a Lévy process. Using the Esscher transform, the Lévy process X in Equation (28) can be considered under a locally equivalent measure such that $e^{-rt}S_t$ is a martingale in its natural filtration $(\mathcal{F}_t)$, which coincides with the natural filtration of X. We assume that P is used as a pricing measure and the market contains no arbitrage opportunities, so that the price of a contingent claim with maturity T is given by the expectation under P of its discounted payoff.

Such a model has been considered in [24], [15], [61] among others. However most of this work is dedicated to abstract valuation of contingent claims and the only example provided is that of vanilla options —except [15] who use analytic techniques and whose results do not cover all cases. Special cases will be studied in Section 4, and include well-known models such as the Normal Inverse Gaussian and the Variance Gamma distribution.

3.2 Difference with Brownian / diffusion models

Before we turn to our main results, let us explain what changes dramatically when Lévy processes are used in financial modelling. Until recently, processes used to model the price of an asset in view of the valuation of derivative instruments were essentially diffusion processes, that is, processes solution to an SDE
$$(29) \qquad\qquad dS_t = b_t dt + \sigma_t dW_t,$$

where W is a Brownian motion. The first such example is the most famous Black-Scholes model [14], where $b_t = b$ and $\sigma_t = \sigma$ are constant over time. However, this model turned out to be unsatisfactory, because market observations highly contradict the hypotheses. In particular, the volatility is not the same for all strikes and maturities of vanilla options, contrary to what the model postulates. Extensions have been investigated, letting b and σ be stochastic processes; the most famous ones are probably Dupire's model, where σ is assumed to be a function of t and S_t, and $b_t = 0$, and, otherwise, stochastic volatility models, where σ is taken to be the solution to another SDE

$$d\sigma_t = \alpha_t dt + \gamma_t dB_t$$

where again B is a Brownian motion, independent or not of W.

Let β_t be the discount factor for date t (i.e. investing \$ β_t today ($t = 0$) will give \$ 1 at time t), most often supposed to be deterministic in equity models. Risk-neutral valuation of derivatives (or: options, or: contingent claims) consists in choosing a probability measure P under which $\beta_t S_t$ is a local martingale, and in computing the price as the expectation under this probability. Let H be the payoff, at time T, i.e. a non-negative, $\mathcal{F}_T$-measurable random variable, the price of an option with payoff H is given by $E_P[\beta_T H]$. When the solution S to (29) is a Markov process (e.g. if $b_t = b(t, S_t)$ and $\sigma_t = \sigma(t, S_t)$ with some regularity), the Feynman-Kac theorem implies that the price process $E_P[\beta_T H/\beta_t | \mathcal{F}_t]$ is a function $u(t, S_t)$, where u solves the partial differential equation associated with the diffusion (29), namely

$$A_t f = 0$$

where A_t is the generator of the process S,

$$A_t f(x) = b(t, x) f'(x) + \frac{1}{2} \sigma(t, x)^2 f''(x), \qquad f \in C^2$$

The situation is even more favorable if one knows the transition semi-group (P_t) of S explicitly, because it is then enough to write

$$E[h(S_T)] = P_T h(S_0).$$

For instance in the Black-Scholes model, the semi-group of S is given by

$$P_t f(s) = \int_0^\infty f(y) e^{-\frac{1}{2\sigma^2 t}(\log(y) - \log(s_t))^2} \frac{dy}{y \sigma \sqrt{2\pi t}}.$$

where $s_t = S_0 e^{(r - \sigma^2/2)t}$. The Feynman-Kac formula holds as well for discontinuous Markov processes, and in particular for Lévy processes. In fact, a Lévy process with Lévy-Khintchine representation (1) has generator

$$Af(x) = a f'(x) + \frac{\sigma^2}{2} f''(x) + \int_{\mathbb{R} - \{0\}} \left(f(x + y) - f(x) - 1_{|y| < 1} y f'(x) \right) \nu(dy);$$

hence the price process of an option with payoff $h(S_T)$ is a function $u(S_t)$, where u solves

$$Au = 0$$

with appropriate boundary conditions. However, contrary to the case of continuous processes, the generator is not a differential operator (operators like A are called integro-differential, or pseudo-differential operators); to solve $Au = 0$, even numerically is more difficult than when A is a differential operator.

The following example illustrates the role of the Lévy measure in option pricing. Consider for instance, in the model described in Section 3.1, a European

call option, with payoff $H = h(S_T) = (S_T - K)^+$. Also assume there are no interest rates, so that the price is given by

$$
\begin{aligned}
E[(S_T - K)^+] &= \int_K^\infty P(S_T > x)dx \\
&= \int_K^\infty P(X_T > \ln(x/S_0))dx \\
&= S_0 \int_k^\infty P(X_T > y)e^y dy
\end{aligned}
$$

where $k = \ln(K/S_0)$ and X is Lévy. Suppose for the sake of the argument that X has no drift, no Brownian component and finite Lévy measure ν. Then, by the compensation formula, we get

$$
\begin{aligned}
P(X_T > y) &= 1_{0>y} + E \int_0^T ds \int_{\mathbb{R}-\{0\}} \nu(dx)(1_{X_s+x>y} - 1_{X_s>y}) \\
&= 1_{y<0} + \int_0^T ds E \int_{-\infty}^0 \nu(dx)1_{x+y<X_s<y} - \int_0^T ds E \int_0^\infty \nu(dx)1_{y<X_s<y+x} \\
&= 1_{y<0} + \int_0^T ds \left(E1_{X_s<y}\nu^-(X_s - y) - E1_{X_s>y}\nu^+(X_s - y) \right)
\end{aligned}
$$

where $\nu^-(z) = \int_{-\infty}^z \nu(dx)$ for $z < 0$ and $\nu^+(z) = \int_z^\infty \nu(dx)$ for $z > 0$ are the tails of ν. This shows the role played by the Lévy measure.

Despite the difficulties underlined above, efficient methods exist to price European options with Lévy processes. Since a Lévy process X is most often known via its Lévy exponent ϕ, one can use Fourier inversion to compute $P(X_T > y)$; the following formula can be found in e.g. Lukacs [45]

$$
P[X_T > y] = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{e^{-iuy}e^{-T\phi(u)} - e^{iuy}e^{-T\phi(-u)}}{iu}du.
$$

Based on these and other considerations, several families of models have been developed that use Lévy processes and derive prices of vanilla options, and general studies have also been made. A theoretical argument is developed in [44] that explains why Lévy processes could remedy some drawbacks of modelling with continuous processes.

Chan [24] examines European option pricing when the underlying price process is modelled as

$$
dS_t = S_{t-}(b_t dt + \sigma_t dY_t)
$$

where Y is a Lévy process satisfying certain technical conditions, and b and σ are deterministic functions. This model is more general than the one we will consider, since S is then the exponential of a process with independent, but not stationary, increments (these processes are called additive processes in [56]). As a choice of a pricing measure, Chan examines a generalized Esscher transform

and the minimal variance measure of Föllmer and Schweizer. He is then able to derive an integro-differential equation for the price process of a European option.

Yan et al. [61] study a model similar to, but slightly more general than the one considered by Chan. They deal with the options pricing problem by using the "numeraire portfolio" approach of Bajeux and Portait [2], thereby going around the problem of choosing a pricing measure. They also derive an integro-differential equation for the price process of a European option, similar to the one of Chan. In the case when S is the exponential of a Lévy process (the same model as in our case), they are able to give an explicit solution by using an approximation with compound Poisson processes.

In this chapter, we restrict attention to the case $S_t = \exp(X_t)$ for a Lévy process X. Several "families" of such models have been studied, beginning with Merton [47]. Although not stated in these terms, Merton's model falls in our framework, as well as the model of Bellamy and Jeanblanc [10] in the case of constant coefficients.

A second family of models is based on Brownian motion, time-changed with an independent subordinator. Barndorff-Nielsen [5] introduces the so-called Normal Inverse Gaussian (NIG) model in finance. The NIG model can be generalized into the Generalized Hyperbolic (GH) model. These distributions have first been studied in [6], while the financial applications have been considered by Eberlein and co-authors [31], [53], [54].

A third family was initiated by Madan and Seneta [46], who study the so-called Variance Gamma (VG) model. This has been further generalized by Carr, Madan, Geman and Yor [32], [21](the CGMY model), and models built on this are currently investigated ([22]).

Let us note that more general models have also been considered, which embed a jump process. Bellamy and Jeanblanc [10] consider the special case when the jump part is a Poisson process with stochastic intensity. Aase [1] proposes a multi-dimensional model where asset price processes are driven by a combination of Ito and point processes (see also [4]). Lastly, we mention that Lévy processes can be used to build stochastic volatility models: Barndorff-Nielsen [8] studies a model where S is driven by a standard SDE

$$dS_t = S_t[rdt + \sigma_t dW_t]$$

and the squared volatility σ_t^2 is a generalized Ornstein-Uhlenbeck process

$$d\sigma_t^2 = -\lambda \sigma_t^2 + dX_t$$

where X is a Lévy process.

However, in this set of works, the main properties being studied are distributional and statistical properties of the asset price process S and the possible definition of a martingale measure. The only option pricing problem which appears reduces to vanilla options.

Some work has been done about the valuation problem for some exotic options. Gerber and Shiu [34] concentrate on perpetual options. They have also

been (among) the first to propose the use of Esscher transforms for the purposes of option pricing. However, their results are limited to the case when the jump part of X is a compound Poisson process (i.e. the Merton model).

Boyarchenko and Levendorskii [15] mimic the techniques used in the case of diffusion processes, where differential operators are at hand, to the case of Lévy processes that induces integro-differential operators, to address the valuation problem for barrier options. Their results encompass a wider range of processes than those of Gerber and Shiu, but are still not fully general.

Here we consider the case when $S_t = \exp(X_t)$, for a general Lévy process X. However, let us note that for the sake of financial modelling, attention should be restricted in practice at least to those processes such that the exponential functional $A_t = \int_0^t S_u du$ makes sense for all t, in order to be able to consider e.g. Asian options. Some properties of A are given in [17, 18, 19], but closed form formulae for the distribution of A are extremely rare. In the case $t = \infty$ (which is of interest in Insurance and ruin theory), a number of results have been obtained by Paulsen [49, 50] and Gjessing and Paulsen [35].

In the framework described above, we take up the problem of the valuation of barrier and lookback options. Using fluctuation theory, we give a method that allows to compute the price of such options in great generality. In particular, our methodology applies to the family of VG models.

3.3   Barrier options

Barrier options have been introduced to remove an unwanted part of the risk present in plain vanilla options. Also, the barrier feature yields a lower price than standard options for an otherwise identical behavior, allowing investors to bet on their beliefs (whether the barrier will be crossed or not). More sophisticated contracts have been built by financial engineering teams that make use of this feature. According to the definition of the contract, we will distinguish between several types of barrier options. For all of them, the final payoff is that of a Call or a Put option, conditionally on the underlying (not) crossing a level specified at the inception of the contract. In the sequel, S will denote the price process of the underlying security. The different kinds of barrier options are given by their payoff, where K is the strike price and H the barrier level:

- Up and In Call: $(S_T - K)^+ 1_{\sup_{t \le T} S_t > H}$;

- Up and Out Call: $(S_T - K)^+ 1_{\sup_{t \le T} S_t < H}$;

- Down and In Call : $(S_T - K)^+ 1_{\inf_{t \le T} S_t < H}$;

- Down and Out Call: $(S_T - K)^+ 1_{\inf_{t \le T} S_t > H}$.

We abbreviate these payoff functions by UIC, UOC, DIC and DOC respectively; these notations will also be used to denote the price of the option. For the corresponding Put option, replace $(S_T - K)^+$ with $(K - S_T)^+$. In order to discard the trivial cases, we will always suppose that $S_0 \vee K < H$ ($S_0 \wedge K > H$) for Up (Down) options. See also [26] for other types of related options.

Before going further, let us point out the following relations between the payoffs of barrier options and vanilla options:

$$\begin{aligned} UIC(K,H) + UOC(K,H) &= Call(K) \\ DIC(K,H) + DOC(K,H) &= Call(K) \end{aligned}$$

which allow us to consider only the family of "In" options. Also, it is clear that the pricing formula for "Down" options can be deduced from the pricing formula for "Up" options by considering the "dual" price process $1/S$ in place of S. Lastly, because of the well known parity relation between Call and Put options, it is enough to deal with Call options. Hence we will give details only for the "Up and In Call" option.

Reduction to Binary options

Having restricted attention to the case of the Up and In Call option, we now show how to reduce the problem of pricing this option to that of pricing a Binary (or: Digital) option.

The payoff of an Up and In barrier Call option is

$$(30) \qquad (S_T - K)^+ 1_{\sup_{t \leq T} S_t > H}.$$

Differentiating this expression with respect to the strike price K, we obtain

$$(31) \qquad -1_{S_T > K} 1_{\sup_{t \leq T} S_t > H}.$$

which is the negative of the payoff of a Binary Up and In Call option with strike price K and barrier H (abbreviated as BinUIC(K,H)). By arbitrage, the same relation must hold between the prices:

Proposition 3.1 *For all K and H*

$$(32) \qquad UIC(K,H) = \int_K^\infty BinUIC(k,H)dk.$$

Note that this relation does not depend on the model under consideration. Similar relations hold for other types of (barrier) options.

3.4   Lookback options

Lookback options have been introduced to keep track of the minimum (or maximum) level reached by the asset price during the time interval of interest. The fixed strike lookback options call and put, have payoff

$$(33) \qquad (\max_{t \leq T} S_t - K)^+ \quad \text{and} \quad (K - \min_{t \leq T} S_t)^+;$$

denoted by $LBCall^{fi}(K)$ and $LBPut^{fi}(K)$; the floating strike lookback call and put have payoff

$$(34) \qquad (S_T - \lambda \min_{t \leq T} S_t)^+ \quad \text{and} \quad (\lambda \max_{t \leq T} S_t - S_T)^+$$

denoted by $LBCall^{fl}(\lambda)$ and $LBPut^{fl}(\lambda)$.

Also of interest are the variable notional call and put options with respective payoffs

$$(35) \qquad \frac{(S_T - \min_{t \leq T} S_t)^+}{\min_{t \leq T} S_t} \quad \text{and} \quad \frac{(\max_{t \leq T} S_t - S_T)^+}{\max_{t \leq T} S_t}$$

however we will restrict interest on the first two types.

To compute the price of fixed strike lookback options, all we need to know is the distribution of the supremum or infimum functional. Let us consider the case of the supremum, the case of the infimum being completely analogous. The distribution of the supremum functional can be deduced from that of the first passage times, since

$$\max_{t \leq T} S_t > x \iff T^S_x \leq T.$$

The distribution of T^S_x is characterized by a special case of the Pecherskii-Rogozin identity, by setting $\beta = 0$ in Eq. (10):

$$\int_0^\infty e^{-qx} E[\exp(-\alpha T^S_{S_0 e^x})] dx = \frac{\kappa(\alpha, q) - \kappa(\alpha, 0)}{q \kappa(\alpha, q)}.$$

Inverting this Laplace transform is then enough to compute the price of the fixed strike lookback call.

Alternatively, we can consider the following strategy for pricing the fixed strike lookback call. We have

$$
\begin{aligned}
(\max_{t \leq T} S_t - K)^+ &= \int_K^{\max_{t \leq T} S_t} dk \\
&= \int_K^\infty 1_{\max_{t \leq T} S_t > k} dk
\end{aligned}
$$

The integrand can be seen as the payoff of a binary Up and In call option with barrier level k and strike price 0. The prices must also satisfy this relationship:

Proposition 3.2 *For all K, it holds that*

$$(36) \qquad LBCall^{fi}(K) = \int_K^\infty BinUIC(0, k) dk$$

We shall now show that the floating strike lookback options can be expressed in terms of Binary barrier options. Consider for instance the floating strike

lookback put option. It is clear that

$$
\begin{aligned}
(\lambda \max_{t \leq T} S_t - S_T)^+ &= \int_{S_T}^{\lambda \max_{t \leq T} S_t} dk \\
&= \int_0^\infty 1_{S_T \leq k \leq \lambda \max_{t \leq T} S_t} dk \\
&= \int_0^\infty 1_{S_T \leq k} 1_{\max_{t \leq T} S_t \geq \frac{k}{\lambda}} dk
\end{aligned}
$$

and by the absence of arbitrage, the prices must stand in the same relationship:

Proposition 3.3 *For all λ, we have*

(37)
$$
LBPut^{fl}(\lambda) = \int_0^\infty BinUIP(k, \frac{k}{\lambda}) dk
$$

Note that if $S_0 \geq \frac{k}{\lambda}$, the barrier feature is meaningless, so that $BinUIP(k, \frac{k}{\lambda})$ = $BinPut(k)$ where $BinPut(k)$ is the price of a binary put option, that pays off $1_{S_T \leq k}$. Hence,

$$
LBPut^{fl}(\lambda) = \int_0^{k/\lambda} BinUIP(k, \frac{k}{\lambda}) dk + \int_{k/\lambda}^\infty BinPut(k) dk.
$$

As we will see later in section 3.6, it is not the most efficient way of pricing options to write them as integrals of appropriate related digital options. However, this is a totally general method and gives some insight on a possible hedging strategy.

3.5 Pricing of the BinUIC option

All that we need now is to compute the price of the BinUIC option. For this purpose we will use the Pecherskii-Rogozin identity (10).

First, rewrite the payoff in terms of the running maximum. The digital barrier option with barrier level H and strike K pays at maturity T:

(38)
$$
1_{S_T > K} 1_{M_T^S > H}
$$

where $M_t^S = \sup_{s \leq t} S_s$. Denoting by M the process $M_t = \sup_{s \leq t} X_s$, $k = \log(K/S_0)$ and $h = \log(H/S_0)$, this payoff can be expressed in terms of the Lévy process X:

(39)
$$
1_{X_T > k} 1_{M_T > h}.
$$

Writing $\sigma = \inf\{t, M_t^S > H\} = \inf\{t, M_t > h\} = T(h)$ with the notation of section 2 the first exit time from the interval $(-\infty, h]$ for the process X, we can also write this payoff as

(40)
$$
1_{X_T > k} 1_{\sigma \leq T}.
$$

Let us now compute the price:

$$
\begin{aligned}
p &= e^{-rT} E[1_{X_T > k} 1_{\sigma \le T}] \\
&= e^{-rT} E[E[1_{X_T > k} 1_{\sigma \le T} | F_\sigma]] \\
&= E[e^{-r\sigma} 1_{\sigma \le T} E[e^{-r(T-\sigma)} 1_{X_T > k} | F_\sigma]]
\end{aligned}
$$

On the event $\{\sigma \le T\}$, the inner conditional expectation is the price of a Binary Call option when the spot is $S_0 e^{X_\sigma}$ and the maturity is $T - \sigma$. Because of the Markov property of the process X, this is a function of (σ, X_σ) which we will denote by $BinCall(\sigma, X_\sigma)$. This type of options can be valued by inverting the Fourier transform of the log of the asset price —in our case, the Lévy process X (see e.g. [37] or [23]). The method consists in decomposing the payoff into appropriate pieces and then performing a change of numéraire on each piece so that the price of the options is expressed through different measures (probabilities) of the exercise set of the option. See subsection 3.6.2 for a direct application of this technique to the case of barrier options.

The case of a European Digital Call option is especially simple, since the price of such an option whith maturity t and strike e^k is

(41)
$$
p_D e^{rt} = E[1_{X_t > k}] = 1 - P[X_t \le k].
$$

It is then immediate to obtain the price by inverting the characteristic function of X:

$$
P[X_t \le k] = \frac{1}{2} + \frac{1}{2\pi} \int_0^\infty \frac{e^{-iuk} e^{-t\phi(u)} - e^{iuk} e^{-t\phi(-u)}}{iu} du
$$

In order to end the computation of the barrier option price, we need the joint law of σ and X_σ. This law is characterized by its Laplace transform given in Theorem 2.1. Mathematically, we have just stated the well-known fact that a distribution is characterized by its Laplace transform. This Laplace transform can be inverted numerically to retrieve the distribution function of (σ, X_σ), which can in turn be used to compute the price of the digital barrier options; hence the integrals giving the price of barrier and lookback options can —at least in principle— be computed numerically. However, due to the dimension of the problem, this method is likely not to be very efficient from a computational point of view, especially if naive algorithms are used. An alternative to the numerical integration required to invert the Laplace transform would be to use an expansion with respect to some appropriately chosen functions.

In the next section we study some special cases when the formulas found before can be simplified —the numerics are then simplified accordingly.

Remark 3.1 *A parallel can be made with the technique employed in [33] to compute the price of barrier options in the Black-Scholes model. They compute the Laplace transform in time (i.e. with respect to the maturity) of the random variable of interest. This amounts to making the maturity an independent exponential time. The same approach can be taken here —and has actually been one of the steps taken to derive the preceding formulas. Let θ be an exponential variable with an exponential distribution with parameter $q > 0$. Excursion*

theory teaches us that the variables M_θ and $X_\theta - M_\theta$ are independent and their Fourier transforms are the Wiener-Hopf factors. So the Laplace transform, with respect to the maturity, of the price of a fixed strike lookback option is readily linked to the Wiener-Hopf factor ϕ^+.

3.6 Simplifications in special cases

As mentioned above, the pricing formulas can be simplified in some cases, leading to reduction of the dimension of the numerical problem. However, to express the prices of options as integrals of elementary payoffs (namely digitals) provides some insight about the hedge of these options.

We first study the case when the background Lévy process has no positive jumps, in which case the overshoot is 0 a.s. Then we show how to use the Esscher transform in order to reduce the numerical cost of computing the price of a barrier option.

3.6.1 The case with no positive jump

As noted in section 2.3, the problem we study is greatly simplified when X has no positive jumps since standard martingale arguments can be used.

Since X has no positive jumps, the barrier level h can only be crossed continuously. Therefore if we still denote by σ the first hitting time of the interval (h, ∞), we have necessarily $X_\sigma = h$. This simple fact allows for huge simplifications.

As noticed in Section 2, the Laplace transform of σ is given by (Equation (20)):

$$E[e^{-\lambda\sigma}] = e^{-h\Phi(\lambda)}$$

where Φ is the inverse bijection of the Laplace exponent of X. The problem has one dimension less than the general case, a significant improvement from a numerical point of view.

Let us turn to the concrete example of a fixed strike lookback option. Proposition 2.3 asserts that if θ is an exponential variable with parameter q, independent from X, then M_θ has an exponential distribution with parameter $\Phi(q)$. Hence,

$$
\begin{aligned}
E[(e^{M_\theta} - e^k)^+] &= \int_0^\infty (e^m - e^k)^+ \Phi(q) e^{-m\Phi(q)} dm \\
&= \Phi(q)\left(\int_k^\infty e^{-m(\Phi(q)-1)} dm - e^k \int_k^\infty e^{-m\Phi(q)} dm\right) \\
&= \Phi(q)\left(\frac{e^{-k(\Phi(q)-1)}}{\Phi(q)-1} - \frac{e^{-k(\Phi(q)-1)}}{\Phi(q)}\right) \\
&= e^{-k(\Phi(q)-1)}\frac{1}{\Phi(q)-1}
\end{aligned}
$$

this formula being true for all q such that $\Phi(q) > 1$, i.e. $q > \psi(1)$.

Hence, the Laplace transform in the maturity of the price of a fixed strike

lookback call option is given by —we denote by k the log-moneyness, $k = \ln(K/S_0)$:

$$\int_0^\infty e^{-qt} LBCall^{fi}(t,K)dt = S_0 \int_0^\infty e^{-(q+r)t} E[(e^{M_t} - e^k)^+]dt$$

$$= \frac{1}{q+r} E[(e^{M_\tau} - e^k)^+]$$

where τ is an exponential variable with parameter $q + r$, and by the foregoing, this is equal to

$$\frac{1}{(q+r)(\Phi(q+r) - 1)} e^{-k(\Phi(q+r)-1)}$$

So in this special case, we only have to invert a one-dimensional Laplace transform. We give in Appendix A some numerical results based on this technique.

3.6.2   Change of numéraire

In order to avoid the integration required to obtain the price of e.g. a Knock-in Up barrier option or a Lookback Put option from the price obtained for the corresponding digital option, one can use a change of measure in order to decompose the expectation we wish to compute into two similar pieces. To do this, we have to make an additional hypothesis on X; however this hypothesis is not really restrictive and is quite reasonable from a financial point of view.

Suppose that $E[e^{X_1}] < \infty$; this implies that X, which represents the logarithm of the returns has moments of all orders, which is quite sensible from a financial point of view. This hypothesis allows us to consider the Esscher transform of P defined by

(42) $$P^X|_{\mathcal{F}_t} = \frac{e^{X_T}}{E[e^{X_T}]} \cdot P|_{\mathcal{F}_t};$$

for all $t > 0$. By Theorem 2.3, X remains a Lévy process under P^X, and its Lévy exponent ϕ^X under P^X is given by

(43) $$\phi^X(u) = \phi(u - i) - \phi(-i)$$

Consider for instance the case of a Knock-in Up barrier call option and suppose for simplicity that $S_0 = 1$ and $r = 0$. Its payoff is

$$(e^{X_T} - K)1_{X_T > k}1_{T(h) \leq T}.$$

Hence the price is given by:

$$E[(e^{X_T} - K)1_{X_T > k}1_{T(h) \leq T}] = E[e^{X_T}1_{X_T > k}1_{T(h) \leq T}] - KP(X_T > k, T(h) \leq T).$$

The last term is exactly the price of a digital knock-in option and can be computed as explained before. We can deal with the first term in the following way. Applying the Esscher transform mentioned above, it can be written as

$$P^X[X_T > k, T(h) \leq T].$$

This can be interpreted as the price of digital option with respect to a different pricing measure. Because X remains a Lévy process under P^X, the same computations as before can be made, leading to a similar result, only with different parameters.

This method can be applied to reduce the computational cost of the numerical computation, since it gets rid of a numerical integration. We have put it here separately because it requires a supplementary hypothesis on the Lévy process X (still keeping a good generality). However the general method developped above, by decomposing the payoff into elementary products (namely digitals) provides some insight on the hedging which is lost in this more efficient method.

4   Examples

In this section we give a few examples where the function κ can be computed in terms of known functions. They include

- the usual geometric Brownian motion;

- a jump-diffusion model, which is a particular case of the model considered by [10];

- The case when X is a subordinator and particularly the Gamma process;

- the case of a NIG model, introduced by [5] and also studied by [15];

- the variance-gamma model introduced by [20], to which the method in [15] does not apply.

4.1   The usual geometric BM

The geometric BM falls into our class of models and therefore we can apply the results developed before. A first approach could be to compute κ from formula (8) (see Proposition 2.1) since when X is a brownian motion with drift ν, we have

(1) $$P(X_t \in dx) = P(N(\nu t, t) \in dx) = g_{\nu t, t}(x)dx$$

where g_{m,s^2} denotes the density of a Gaussian variable with mean m and variance s^2.

Now since X has continuous paths, and a *fortiori* no positive jumps, the simplified approach of section 3.6.1 can be used. Indeed, this amounts to applying the well-known reflection principle for brownian motion. Specializing equation (20), we recover the well-known formula for the hitting times of Brownian motion with drift ν:

$$E[e^{-\lambda T(x)}] = e^{-x(-\nu + \sqrt{\nu^2 + 2\lambda})}.$$

The change of numéraire technique in this setting is also well-known thanks to the Cameron-Martin formula. In fact, S remains a geometric brownian motion

under the new probability, with the only difference that its drift coefficient is modified. Using this and the explicit knowledge of the semigroup and resolvent operators for the Brownian motion, [33] have obtained very explicit formulas for the Laplace transform in time of the price of a barrier option; specializing their results to $L = 0 \Leftrightarrow a = +\infty$ (with their notation):

$$
\begin{aligned}
\Phi(\theta) &= E \int_0^\infty e^{-\theta u}(S_u - K)^+ 1_{T_H^S \leq u} du \\
&= e^{-\mu b} g_1(e^b)
\end{aligned}
$$

where

$$
g_1(e^b) = 2 \left\{ \frac{e^{b(\nu+1)}}{\mu^2 - (\nu+1)^2} - \frac{Ke^{b\nu}}{\mu^2 - \nu^2} \right\} + \frac{e^{-\mu b}K^{\nu+1-\mu}}{\mu(\mu+\nu)(\mu+\nu+1)},
$$

$b = \log(H/S_0)$ and $\mu^2/2 = \theta + \nu^2/2$.

4.2 Jump-diffusion

Here we consider a special case of the model studied by [10]. Let the process S be given by the SDE

$$
(2) \qquad\qquad dS_t = S_{t-} [\mu dt + \sigma dW_t + \zeta dN_t]
$$

where W is a standard Brownian motion, N is a Poisson process with constant intensity λ and $\zeta > -1$. The drift μ is chosen so that $e^{-rt}S_t$ is a martingale:

$$
\mu = r - \frac{\sigma^2}{2} - \lambda\zeta
$$

It is well known that S is the Doléans-Dade exponential, multiplied by a drift term:

$$
\begin{aligned}
S_t &= S \exp \left[\left(r - \frac{\sigma^2}{2} - \lambda\zeta \right) t + \sigma W_t + \ln(1+\zeta)N_t \right] \\
&= Se^{X_t}
\end{aligned}
$$

where X is the Lévy process

$$
(3) \qquad\qquad X_t = \left(r - \frac{\sigma^2}{2} - \lambda\zeta \right) t + \sigma W_t + \ln(1+\zeta)N_t.
$$

Choose first $\zeta > 0$, and according to Section 2.3, let ψ be the Laplace exponent of $-X$, $E[e^{-uX_t}] = e^{t\psi(u)}$; we have

$$
(4) \qquad \psi(u) = - \left(r - \frac{\sigma^2}{2} - \lambda\zeta \right) u + \frac{1}{2}\sigma^2 u^2 - \lambda \left(1 - (1+\zeta)^{-u} \right).
$$

Let Φ be the inverse of ψ; according to Theorem 2.2, the ladder exponent κ is given by

$$(5) \qquad \kappa(\alpha, \beta) = c \frac{\alpha - \psi(\beta)}{\Phi(\alpha) - \beta}.$$

The Laplace transform of the pair $(T(x), X_{T(x)})$ is then given by

$$\int_0^\infty e^{-qx} E[e^{-\alpha T(x) - \beta(X_{T(x)} - x)}] dx =$$

$$(6) \qquad \frac{1}{q - \beta} \left\{ 1 - \frac{(\alpha - \psi(\beta))(\Phi(\alpha) - q)}{(\Phi(\alpha) - \beta)(\alpha - \psi(q))} \right\}$$

The price of a Up and In Call option with strike price K, barrier H and maturity T is given by

$$(7) \qquad BinUIC(K, H, T) = \int_0^T \int_h^\infty BinCall(Se^y, K, T - t) dF(t, y)$$

where $h = \ln(H/S_0)$ and F is the joint distribution function of $(T(x), X_{T(x)})$, that can be obtained from the above Laplace transform.

If we choose $\zeta < 0$, the results of section 3.6.1 apply. Alternatively if we try to follow the steps of [33], we find that the resolvent operators are given by

$$
\begin{aligned}
V_\theta \phi(x) &= E\left[\int_0^\infty e^{-\theta u} \phi(X_u) du | X_0 = x \right] \\
(8) \qquad &= \frac{1}{\zeta\sqrt{\sigma}} \int dy \phi(y) \sum_{n=0}^\infty \frac{1}{n!} \left(\frac{\lambda\sigma}{\zeta^2} \right)^n e^{-\frac{\mu x n}{\zeta}} \tilde{K}_{n+\frac{1}{2}}(z_n(x, y))
\end{aligned}
$$

with $\zeta = \sqrt{2\sigma^2(\theta + \lambda) + \mu^2}$, $z_n(x, y) = \frac{\sqrt{\zeta}}{\sigma^2}(n \ln(1 + \zeta) - x - y)$, K_ν are the Macdonald functions and $\tilde{K}_\nu(x) = \sqrt{2/\pi} x^\nu K_\nu(x)$. The complexity of the expression of the resolvent operators makes it impossible to carry on the programme worked out in [33].

Remark 4.1 *In the case of the Merton model, which generalizes the jump-diffusion model studied above, Kou and Wang [41] have obtained some results about $T(x)$ when*

$$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$$

and the Y_i have density

$$p\eta_1 e^{-\eta_1 y} 1_{y > 0} + (1 - p)\eta_2 e^{-\eta_2 y} 1_{y < 0}$$

for some $0 \leq p \leq 1$, $\eta_1 > 0$, $\eta_2 > 0$.

4.3 The case of a subordinator

We consider the case when X is a subordinator. Although not very interesting in practice, this simple case sheds light on our study. When X is a subordinator, the space integral in formula 8 has a very simple expression in terms of the Lévy exponent of X. Indeed,

$$\int_0^\infty e^{-\beta x} P(X_t \in dx) = e^{-t\phi(i\beta)}.$$

Hence κ is simply given by

$$\kappa(\alpha, \beta) = k \exp\left(\int_0^\infty t^{-1}\left(e^{-t} - e^{-t(\alpha+\phi(i\beta))}\right) dt\right)$$

and this can be evaluated by the Frullani integral:

$$\kappa(\alpha, \beta) = k \exp\left(\ln\left(\alpha + \phi(i\beta)\right)\right) = k(\alpha + \phi(i\beta)).$$

Note that because X is a subordinator, we have $\hat{\kappa} \equiv 1$ and the Wiener-Hopf factorization (formula (16)) reduces to the equality above.
In particular if X is a Gamma process, whose characteristic exponent is given by

$$\phi(u) = \frac{\mu}{\nu}\ln\left(1 - iu\frac{\nu}{\mu}\right),$$

we find that

(9) $$\kappa(\alpha, \beta) = k\left(\alpha + \frac{\mu}{\nu}\ln\left(1 + \beta\frac{\nu}{\mu}\right)\right).$$

We now turn to the cases when the 1-dimensional laws of X are known explicitly. In such cases, formula (8) can be used to compute the bi-variate Laplace transform κ. In principle, this is already possible as soon as the Lévy exponent is known explicitly, since then the density (if it exists) can be computed by Fourier inversion. The cases of real interest though are those when the density itself is known explicitly, since then the integral in (8) may have a nice form. Among others are:

1. the Normal Inverse Gaussian (NIG) model;

2. the Variance-Gamma (VG) model.

4.4 Normal Inverse Gaussian

The NIG distribution with parameters $(\alpha, \beta, \delta, \mu)$, introduced by [5] has density

(10) $$d_{NIG}(x) = \frac{\alpha}{\pi} e^{\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)} \frac{K_1\left(\alpha\delta\sqrt{1+\left(\frac{x-\mu}{\delta}\right)^2}\right)}{\sqrt{1+\left(\frac{x-\mu}{\delta}\right)^2}},$$

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where K_1 is a modified Bessel function; the corresponding characteristic function is:

(11)
$$\phi_{NIG}(u) = e^{i\mu u} \frac{e^{\delta\sqrt{\alpha^2 - \beta^2}}}{e^{\delta\sqrt{\alpha^2 - (\beta + iu)^2}}}.$$

The increments of length h of the Lévy process with ϕ_{NIG} as a Lévy exponent have a NIG distribution with parameters $(\alpha, \beta, \delta h, \mu h)$; this means that X_t has distribution $NIG(\alpha, \beta, \delta t, \mu t)$. In order to apply the results of Section 3 we need to compute the integral from formula (8):

$$\int_0^\infty dt \int_0^\infty dx\, t^{-1}(e^{-t} - e^{-at-bx}) d_{NIG}(x)$$
$$= \int_0^\infty dt\, t^{-1} \left(e^{-t} P(X_t \geq 0) - e^{-at} \int_0^\infty e^{-bx} d_{NIG}(x) dx \right)$$

for $a, b > 0$. Using the following integral representation for the function K_1 (see [43]):

$$K_1(x) = \frac{x}{4} \int_0^\infty e^{-z - \frac{x^2}{4z}} z^{-2} dz$$

we find after a few manipulations that

(12)
$$\int_0^\infty e^{-bx} \frac{K_1\left(\alpha\delta\sqrt{1 + \left(\frac{x-\mu}{\delta}\right)^2}\right)}{\sqrt{1 + \left(\frac{x-\mu}{\delta}\right)^2}} dx$$
$$= \frac{e^{-b\mu}\delta\sqrt{\pi}}{2} \int_0^\infty y^{-3/2} e^{\left(\frac{b^2}{\alpha^2} - 1\right)y - \frac{\alpha^2\delta^2}{4y}} N\left(\frac{2by - \alpha^2\mu}{\alpha\sqrt{2y}}\right) dy$$

where N stands for the cumulative distribution function of the standard Gaussian law. Alternatively, the integral in the right side of Eq. (12) can be written as

$$\frac{\sqrt{2\pi}}{\alpha\delta} e^{\sqrt{2}\alpha\delta\left(\frac{b^2}{\alpha^2} - 1\right)} P[\tilde{W}_1 \leq \frac{2bH_z - \alpha\mu^2}{\alpha\sqrt{2H_z}}]$$

where H_z is the first hitting tine of $z = \frac{\alpha\delta}{\sqrt{2}}$ by a Brownian Motion W with drift $m = -2\left(\frac{b^2}{\alpha^2} - 1\right)$ and $\tilde{W}$ is a Brownian Motion independent of W.

The probability $P(X_t \geq 0)$ can be computed as

$$P(X_t \geq 0) = E\left[N\left(-\frac{\mu t}{\sqrt{H_t}}\right)\right]$$

where H is the process of first hitting times for a Brownian motion with drift α and variance δ^2. Note that the preceding approach does not apply to the more general Generalized Hyperbolic model (see, e.g. [31]) because the law of X_t is not known explicitly for $t \neq 1$.

4.5 The Variance-Gamma model

Let b be brownian motion with drift

$$(13) \qquad\qquad b(t; \theta, \sigma) = \theta t + \sigma W_t$$

and $\gamma = \gamma(t; \mu, \nu)$ a Gamma process independent of W. γ is a Lévy process whose increment $\gamma(t+h; \mu; \nu) - \gamma(t; \mu; \nu)$ has density

$$(14) \qquad f_h(g) = \left(\frac{\mu}{\nu}\right)^{\frac{\mu^2 h}{\nu}} \frac{1}{\Gamma(\frac{\mu^2 h}{\nu})} e^{-\frac{\mu}{\nu} g} g^{\frac{\mu^2 h}{\nu} - 1}, \qquad g > 0.$$

The characteristic function of the process γ is given by

$$(15) \qquad \phi_{\gamma(t)}(u) = E[e^{iu\gamma(t;\mu;\nu)}] = \left(\frac{1}{1 - iu\frac{\nu}{\mu}}\right)^{\frac{\mu t}{\nu}}$$

and its Lévy measure is $k_\gamma(x)dx$, where

$$(16) \qquad k_\gamma(x) = \frac{\mu^2}{\nu x} e^{-\frac{\mu}{\nu} x}, \qquad x > 0.$$

The VG process is then defined as

$$(17) \qquad X(t; \sigma, \nu, \theta) = b(\gamma(t; 1, \nu), \theta, \sigma).$$

Alternatively, the VG process can be expressed as the difference of two independent Gamma processes.

Hence the VG process is a Lévy process with no drift and no Gaussian component. Its Lévy measure has density

$$(18) \qquad k(x) = \frac{1}{\nu |x|} e^{\frac{\theta x}{\sigma^2}} e^{-|x| \sqrt{\frac{2}{\nu} + \frac{\theta^2}{\sigma^2}}}.$$

For each t, the law of $X(t)$ has the density given by

$$
\begin{aligned}
f_{X(t)}(x) &= \int_0^\infty \frac{1}{\sigma\sqrt{2\pi g}} e^{-\frac{(x-\theta g)^2}{2\sigma^2 g}} \frac{1}{\nu^{\frac{t}{\nu}} \Gamma(\frac{t}{\nu})} g^{\frac{t}{\nu}} e^{-\frac{g}{\nu}} dg \\
&= \frac{1}{\sqrt{2\pi}\sigma\,\Gamma(t/\nu)} \nu^{\frac{-t}{\nu}} e^{\frac{\theta x}{\sigma^2}} \left(\frac{2}{\nu} + \frac{\theta^2}{\sigma^2}\right)^{-\frac{\nu+2t}{4\nu}} \\
(19) & \qquad\qquad \left(\frac{\sigma^2}{x^2}\right)^{-\frac{\nu+2t}{4\nu}} K_{-t/\nu-1/2}\left(\sqrt{\frac{x^2}{\sigma^2}}\sqrt{\frac{2}{\nu} + \frac{\theta^2}{\sigma^2}}\right)
\end{aligned}
$$

and characteristic function

$$(20) \qquad \phi_{X(t)}(u) = E[e^{iuX(t)}] = \left(\frac{1}{1 - i\theta\nu u + \frac{\sigma^2 \nu}{2} u^2}\right)^{\frac{t}{\nu}}.$$

In [46], [20], [23] the underlying stock is modelled as $S_t = S_0 e^{X_t}$ where X is a VG process. The general results we have derived apply to this situation and all we need now is to compute the bivariate Laplace exponent $\kappa(\alpha, \beta)$. By equation (8), this amounts to computing the integral

$$\int_0^\infty dt \int_0^\infty \frac{1}{t}(e^{-t} - e^{-\alpha t - \beta x}) f_{X(t)}(x) dx.$$

Here again we can express $P(X_t \geq 0)$ in terms of known functions. By using Fubini's theorem, it can be shown that

$$P(X_t \geq 0) = \frac{1}{2} + \frac{1}{\Gamma(t/\nu)} \int_0^\infty e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \Gamma_i^c \left(\frac{\sigma^2 x^2}{\theta^2 \nu}, \frac{t}{\nu} \right)$$

if $\theta \geq 0$, while

$$P(X_t \geq 0) = \frac{1}{\Gamma(t/\nu)} \int_0^\infty e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \Gamma_i \left(\frac{\sigma^2 x^2}{\theta^2 \nu}, \frac{t}{\nu} \right)$$

if $\theta < 0$, where Γ_i is the incomplete Gamma function,

$$\Gamma_i(x, z) = \int_0^x e^{-u} u^{z-1} du$$

and Γ_i^c is the complement $\Gamma_i^c(x, z) = \Gamma(z) - \Gamma_i(x, z)$.

4.6   Stable processes

Stable processes are those Lévy processes which possess, just as the Brownian motion, the *scaling property*: for some $\alpha \in]0, 2]$

(21)
$$\forall k > 0, (X_{kt}, t \geq 0) \stackrel{(law)}{=} (k^{1/\alpha} X_t, t \geq 0).$$

This property implies that $P(X_t > 0)$ does not depend on t; the common value ρ is called the positivity parameter. Formula (2.7) then reduces to

$$\kappa(\alpha, \beta) = k \exp \left\{ \int_0^\infty \frac{dt}{t} \left(\rho e^{-t} - e^{-\alpha t} \int_0^\infty e^{-\beta x} P(X_t \in dx) \right) \right\}.$$

Unfortunately, except in the special cases of the Brownian motion and its hitting times, and the Cauchy process, the law $P(X_t \in dx)$ is not known explicitly, although a lot of information can be found in [30] and [63].

However, some quantities related to the fluctuation theory have simple expression in the case of stable processes. It is shown in [55] that

(22)
$$P(X_\sigma \in dy) = \frac{\sin(\pi\alpha/2)}{\pi} \frac{1}{y} \left(\frac{x}{x-y} \right)^{\alpha/2} dy, \quad y > x$$

for a symmetric stable process, where we recall $\sigma = \inf\{t : X_t > x\}$.

In the case $\rho + k = l/\alpha$ for some integers k and l, Doney [28] shows that the Wiener-Hopf factors are determined in the following way. First,

$$\phi_1^+(u) \;=\; \frac{f_{k-1}(\alpha, (-1)^l(-iu)^\alpha)}{f_{l-1}(1/\alpha, (-1)^{k+1}iu)} \quad u \notin i\mathbb{R}_-$$

(23)

$$\phi_1^-(u) \;=\; \frac{f_{l-1}(1/\alpha, (-1)^{k+1}iu)}{f_k(\alpha, (-1)^l(iu)^\alpha)} \quad u \notin i\mathbb{R}_+$$

with

$$f_m(\varepsilon, u) = \prod_{r=0}^{m}(u + e^{i\varepsilon(m-2r)\pi}), \quad f_{-1}(\varepsilon, u) = 1.$$

Now if θ is an exponential variable with parameter $q > 0$, $q\theta$ is an exponential variable with parameter 1. Thanks to the scaling property of X, we get:

$$
\begin{aligned}
\phi_q^+(u) \;&=\; E[e^{iuM_\theta}] \\
&=\; E[e^{iuM_{q\theta/q}}] \\
&=\; E[e^{iuq^{-1/\alpha}M_{q\theta}}] \\
&=\; \phi_1^+(uq^{-1/\alpha}).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\phi_q^-(u) \;&=\; E[e^{iu(X_\theta - M_\theta)}] \\
&=\; \phi_1^-(uq^{-1/\alpha}).
\end{aligned}
$$

5 Conclusion

In this paper we have shown how the theory of fluctuations can be applied to derive the price of barrier and lookback options when the underlying security is modelled as the exponential of a Lévy process. The general method we used also gives some insight on a possible hedging strategy. Special cases have been studied, where more efficient methods can be used. Some examples have been studied and the general formulas have been simplified to some extent.

We note that similar results, that also rely on the Wiener-Hopf decomposition, have been obtained in [15], using purely analytic techniques; the probabilistic methods used in this paper give completely general results.

A drawback of our method is the complexity of numerical calculations required, since in general numerical n-fold integrals ($n = 3, 4$) are needed. In addition, computational cost may arise when computing the inverse of the Laplace exponent of the Lévy process: this inverse does not necessarily have a closed expression and may need to be evaluated numerically. In such cases, because

the integrand requires itself numerical analysis, it is not clear whether inverting the Laplace transform is better than using e.g. Monte Carlo techniques. We hope to be able to simplify further these formulas and obtain more explicit expressions, so as to reduce the computational drawbacks of our method.

A   Numerical example

In order to illustrate the method discussed in this paper, we provide in this appendix a numerical example, and compare the results with those which come out from the standard Black-Scholes model. We shall examine the case with no positive jumps —as mentioned earlier this reduces the computational complexity.

We consider the jump-diffusion model studied in Section 4.2:

$$X_t = \left(r - \frac{\sigma^2}{2} - \lambda\zeta \right) t + \sigma W_t + \ln(1 + \zeta)N_t$$

for a Brownian motion W and an independent Poisson process N with intensity λ. We make the following choice for the parameters: $r = 0.03$, $\sigma = 0.2$, $\lambda = 0.1$ and $\zeta = -0.3$. These parameters are representative of parameters implied from equities markets today. The Laplace exponent of X is given by

$$\psi(u) = u\left(r - \frac{\sigma^2}{2} - \lambda\zeta \right) + \frac{1}{2}u^2\sigma^2 - \lambda(1 - (1 + \zeta)^u) \quad u > 0$$

We compare the results with those obtained in the Black-Scholes model, namely by considering a Lévy process

$$Y_t = \left(r - \frac{\tilde{\sigma}^2}{2} \right) t + \tilde{\sigma}\tilde{W}_t$$

In order the comparison to be significant, we choose $\tilde{\sigma}$ so that the two processes X and Y have the same predictable quadratic variation; namely

$$\tilde{\sigma}^2 = \sigma^2 + \lambda(\ln(1 + \zeta))^2$$

A.1   Hitting times

First we compute the hitting times probabilities when there are no positive jumps, i.e. $P[T(x) \leq t]$. Recall that for $x > 0$, $T(x) = \inf\{t, X_t > x\}$, and that in the case when $\zeta < 0$, we have $X_{T(x)} = x$ a.s. To compute the distribution function of $T(x)$, we inverted its Laplace transform by numerically evaluating the Bromwich integral: let $\mathcal{L}f(q)$ be the Laplace transform of a function f, such that f and $\mathcal{L}f$ are integrable, then f is given by

$$f(t) = \frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{ts}\mathcal{L}f(s)ds$$

where γ is chosen so that all the singularities of $\mathcal{L}f$ lie in the half plane $\{\mathrm{Re}(z) < \gamma\}$. To compute the distribution function of the first hitting times of Y, we integrate their density, which is given by

$$P[T^Y(x) \in dt] = \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{1}{2t}(x-\nu t)^2} dt$$

with, of course, $T^Y(x) = \inf\{t, Y_t/\tilde{\sigma} = x\}$, $x > 0$ and $\nu = r/\tilde{\sigma} - \tilde{\sigma}/2$. Results are reported in Table A.1.

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t	Jump Diffusion	Black-Scholes
1 month	0.002	0.006
2 months	0.030	0.052
3	0.080	0.114
6	0.228	0.265
9	0.334	0.364
1 year	0.410	0.433
1.5 years	0.509	0.524
2	0.572	0.582
2.5	0.617	0.623
3	0.650	0.655
3.5	0.677	0.680
4	0.698	0.700
4.5	0.716	0.717
5	0.731	0.731
5.5	0.744	0.744
6	0.755	0.755
6.5	0.765	0.765
7	0.774	0.774
7.5	0.782	0.781
8	0.789	0.789
8.5	0.796	0.795
9	0.802	0.801
9.5	0.808	0.807
10	0.813	0.812

Table 1: Probability of hitting the level H, $H/S_0 = 1.2$ (so $x = \ln(1.2)$) before t, for jump-diffusion and Black-Scholes models

A.2 Lookback options

As a second example, we compare the price of fixed strike lookback call options in the Jump-diffusion model and in the Black-Scholes model. We consider the case when $\zeta < 0$ and invert the Laplace transform in time again thanks to the Bromwich integral in order to obtain option prices. Recall from Section 3.6.1 that the Laplace transform of the price of the fixed strike lookback call option, with strike price K is given by

$$q \mapsto \frac{1}{(q+r)(\Phi(q+r)-1)} e^{-k(\Phi(q+r)-1)}$$

where $k = \ln(K/S_0)$, provided $\Phi(q+r) > 1$. Hence, we use the Bromwich integral only for the case $k > 0$. When $k \leq 0$, since $M_t \geq 0$ a.s., we simply have

$$LBCall^{fi}(t,k) = e^{-rt} \left(E[e^{M_t}] - e^k \right)$$

The results are summarized in Table A.2. We use the same values of the parameters as in the previous example.

Maturity	Strike (% of S_0)	Price (% of S_0)	
		Jump Diffusion	Black-Scholes
	80	28.6140	29.7317
	90	18.6887	19.8070
3 months	100	8.7634	9.8823
	110	2.4097	3.0844
	120	0.4501	0.7079
	80	32.8369	34.0523
	90	22.9858	24.2024
6 months	100	13.1347	14.3525
	110	5.6975	6.7485
	120	2.0728	2.7927
	80	38.8138	40.3247
	90	29.1094	30.6202
1 year	100	19.4049	20.9158
	110	11.9472	11.4414
	120	6.6501	6.2641
	80	47.8724	49.7077
	90	38.4547	40.2900
2 years	100	29.0371	30.8724
	110	20.7653	22.5405
	120	9.0224	16.1850

Table 2: Prices of fixed strike lookback call option, in the jump-diffusion and the Black-Scholes model

B Extensions

We have established the Wiener-Hopf factorization for Lévy processes. However, it appears from empirical works that Lévy processes may not be suitable for the modeling of financial data series. In particular, the implied volatility surface from Lévy processes is not consistent with those which are observed for many stocks and indices.

Hence, in order to give our technique a wider range of applicability, it would be desirable to extend it to other processes, in particular:

1. *additive processes*, i.e. processes with independent, but not stationary, increments;

2. Lévy processes, time-changed with an independent increasing process.

The second case is in some sense a particular instance of the first one, since by conditioning on the time-change process, we find ourselves left with a process that has independent, but not necessarily stationary, increments.

B.1 Extension to additive processes

In the first case, the difficulty comes from the fact that the processes we deal with are not homogeneous in time. However, some work can be done, but the results we get are far less explicit than in the case of Lévy processes. We only outline them here.

Specifically, let X be an additive process. Thus, X is a *non-homogeneous* Markov process. Define as before the reflected process $\tilde{M}_t = M_t - X_t$, where $M_t = \sup_{s \leq t} X_s$. Then, it is easy to see, thanks to the independence of the increments of X, that $\tilde{M}$ enjoys the *simple* Markov property (in the filtration of X). But $\tilde{M}$ is not homogeneous, and does not enjoy the *strong* Markov property in the filtration of X. Hence excursion theory cannot be applied directly.

However, as is well-known, the time-space process $Z_t = (t, \tilde{M}_t)$ is a homogeneous Markov process. In fact, as we end up working jointly in time and space already in the case of a Lévy process, it is reasonable to think that we will be able to derive some results also in the present case, by working directly on the time-space process. However, all the properties will have to be "translated" in a convenient manner.

We first note that the semi-group of $(t, \tilde{M}_t)$ is a Feller semi-group (this can be shown in much the same way as [11, Prop. VI.1]), so that Z possesses the strong Markov property[1].

Next, consider the set $J = \mathbb{R}_+ \times \{0\}$. This is a closed set, with empty interior in $\mathbb{R}_+ \times \mathbb{R}_+$, the state space of the time-space process. We make the hypothesis

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[1] This can be stated another way: $\tilde{M}$ enjoys a kind of "non-homogeneous strong Markov property". Retaining the notations in [27], $E_a^\mu[f \circ \theta_T] = F(T, X_T)$ for any finite stopping time T, any starting measure μ, any time starting point a, and any bounded $\mathcal{F}_\infty$ measurable f, where $F(t, x) = E_{a+t}^x[f]$

that every point in J is regular for J, namely

(24) $\qquad \forall x = (t,0) \in J \quad P_x[\inf\{u > 0, Z_u \in J\} = 0] = 1.$

In other words, for all $t \geq 0$, $P[\tilde{M}_t = 0$ and $\inf\{u > t : \tilde{M}_u = 0\} = 0] = 1$. This corresponds to the assumption that 0 is regular for itself in the Lévy case. We also assume that

(25) $\qquad P_x[\inf\{u > 0, \tilde{M}_u > 0\} = 0] = 1$

for any $x \in J$, which corresponds to the fact that 0 is an instantaneous point in the homogeneous case. Note that according to Blumenthal's 0-1 law, each of the probabilities in (24) or (25) is either 0 or 1. If (24) or (25) is not satisfied, the sucessive times at which Z returns in J form a discrete sequence T_n; this case will be examined separately.

Hence, according to [27, Chap. XV, p.273], it is possible to define a local time process for J, i.e. an increasing continuous additive functional L of Z, such that the support of the measure dL is exactly J. We now want to study the process Z in the local time scale.

The inverse local time $\tau_t := \inf\{u > 0, L_u > t\}$ is a process with independent increments, because of the additivity of L. However, τ does not have *stationary* increments. We can define the ladder height process $H = S \circ \tau = X \circ \tau$, and show, just like in [11], that the bivariate process (τ, H) is additive. It follows that there exists a family of functions κ_u such that

$$E[e^{-\alpha \tau_u - \beta H_u}] = e^{-\kappa_u(\alpha, \beta)},$$

and each function κ_u has a Lévy-Khintchine representation

(26) $\quad \kappa_u(\alpha, \beta) = \delta^\tau(u)\alpha + \delta^H(u)\beta + \int_{(0,\infty)\times(0,\infty)} (1 - e^{-\alpha t - \beta h})\nu_u^l(dt, dh).$

We can then follow the same lines as we did for Lévy processes in the proof of the Pecherskii-Rogozin identity 2.1. Lemmas 2.1 to 2.3 remain true; however, the formula in Lemma 2.3 must be amended as

$$dx 1_{H(\eta_x) = x} \delta^H(\eta_x) d\eta_x.$$

This is already a clue that formulas will not be as nice as for Lévy processes. In fact, by following exactly the same lines as in the proof of Theorem 2.1, we obtain:

(27) $\int_0^\infty dx\, e^{-qx} E[e^{-\alpha T(x) - \beta K(x)}] = \int_0^\infty du\, e^{-\kappa_u(\alpha, q)} \left(\frac{\kappa_u(\alpha, q) - \kappa_u(\alpha, \beta)}{q - \beta} \right)$

where we recall $T(x)$ denotes the first hitting times of X and K is the overshoot at x: $K(x) = X_{T(x)} - x$. This formula is in principle the same as formula (10); however, we do not know at the present time a formula similar to (8) that would enable us to actually compute the functions κ_u.

B.1.1 The case of irregular or holding points

Here we enter the world of inhomogeneous Poisson processes.

Let us first suppose that $\inf\{u > 0 : \tilde{M}_u = 0\} > 0$, P_x-a.s. for some $x \in J$. Then the Markov property entails that this holds for any $x \in J$. It follows that the set of times at which $\tilde{M}$ visits J is discrete.

Suppose now that when it visits J, $\tilde{M}$ is "held" there for some time θ. Then θ has an exponential distribution –whose parameter depends on the time when $\tilde{M}$ arrived in J.

B.2 Time changes of Lévy processes

In this paragraph, we deal with what we name time changes of Lévy processes, that is, models of the sort $S_t = Se^{X_t}$ where the process X is taken to be $X_t = Y_{C_t}$, where Y is a Lévy process and C is an increasing process, independent of Y. Such models are discussed in e.g. [22].

The results of the previous paragraph could be applied to the present case, since *conditionnally on* $(C_t, t \geq 0)$, X is an additive process. However, under a slight assumption on C, we are able to get far more explicit and useful results.

Assume that almost every path of C is continuous and strictly increasing. Then $T(x) = C_{T^Y(x)}$ as is easily seen, and as a consequence

(28) $$E[h(T(x), X_{T(x)})] = E[h(C(T^Y(x)), Y_{T^Y(x)})]$$

for any measurable and bounded function h. Hence, since we know the joint distribution of $(T^Y(x), Y_{T^Y(x)})$, we also know the joint law of $(T(x), X_{T(x)})$, and our results extends straightforwardly to the present case.

This method allows to treat the model of [22], where Y is a CGMY (or another Lévy) process, and C is the integrated square-root process:

$$C_t = \int_0^t v_s \, ds,$$

$$dv_s = \kappa(\theta - v_s)ds + \sigma\sqrt{v_s}dW_s$$

with W a Brownian motion, independent of Y.

However, unfortunately, this does not extend to subordination (i.e. cases where C is a subordinator), since there is no continuous subordinator.

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