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REASONABLE EXTREME-BOUNDS ANALYSIS*

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Leamer (1983) has given bounds for a parameter of a model estimated by ordinary least squares for all possible specifications with a given group of explanatory variables. However, some of these specifications will have low R^2 specification and these can lead to wide bounds. In this paper, bounds are derived for all specifications with R^2 values a given percentage of the maximum R^2 value. These exact bounds can be found from calculating only two regressions. The techniques are applied to a study of the velocity of money.

1. Introduction

A modeller is faced with many different possible specifications for the model when there are several possible explanatory variables, each of which can enter with various lags. Leamer (1983) has suggested that certain essential features of the model can vary greatly between alternative specifications, thus making the interpretation of the model difficult or 'fragile'. An easily understood version of his argument has a single dependent variable y, a group of variables $\boldsymbol{X}_{\mathrm{F}}$ that 'should' be used as explanatory variables in any model of y when a particular question is under consideration, and a second group of variables $\boldsymbol{X}_{\mathrm{D}}$ that may or may not enter the model as explanatory variables. A basic or 'restricted' model would be

$$y = \boldsymbol{\beta}_{\mathrm{F}}\boldsymbol{X}_{\mathrm{F}} + \text{residual}, \tag{1}$$

a complete model would be

$$y = \boldsymbol{\beta}_{\mathrm{F}}\boldsymbol{X}_{\mathrm{F}} + \boldsymbol{\beta}_{\mathrm{D}}\boldsymbol{X}_{\mathrm{D}} + \text{residual}, \tag{2}$$

with some linear constraints on $\boldsymbol{\beta}_{\mathrm{D}}$, such as requiring that certain variables be given a zero coefficient in (2). Thus, for example, if one was interested in the effect of an interest rate on velocity, then the equation for velocity would

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certainly include this interest rate in X_F plus possibly also velocity lagged once, money lagged once, and a price index lagged. These explanatory variables might be thought of as a minimum set of variables necessary to explain velocity, according to some theory. This would give the basic model. However, it may be thought necessary, by some modellers, to augment the basic model so that the model better explains the main features of the actual velocity series. This augmentation may include further lags of the variables already used plus other variables, such as variability of money base. As there are many possible ways to augment the basic model so there will be many different specifications. Suppose that we are most interested in the value of a particular coefficient, denoted β_0, such as the coefficients on interest rate in the velocity equation. The estimates of β_0 may vary considerably from one specification to another, and the extremes taken by the alternative estimates are called the 'extreme bounds' by Leamer (1983). The extent of these bounds are viewed as measuring the fragility of the estimate of β_0 as alternative specifications are used. The value and interpretation of these bounds have been strongly criticized by McAleer, Pagan, and Volker (1983, 1985) and also by Breusch (1985) and defended by Leamer (1985).

One criticism of the use of extreme bounds, which has some impact, is that the actual extremes may come from models that most economists would find unreasonable in some way, such as having low Durbin–Watson statistics, for example, in a time-series context. One way to express this problem is in terms of R^2 statistics. We are not defending R^2 as an ideal measure of the quality of a model but it is possibly a relevant statistic and some exact results are achievable using it. Suppose that the maximum value achievable for R^2 is $R^2_{\max}$, which is certainly found by using all of the variables X_F, X_D in (1) with no exclusions, which might be called the 'full' model. Of course, other specifications may also achieve this $R^2_{\max}$. The above worry about the virtues of using extreme-bounds analysis is that the extreme may come from specifications that achieve R^2 values very much smaller than $R^2_{\max}$, and these specifications might be considered irrelevant because of their relatively low goodness-of-fit so that estimates of β_0 based on them would also be strongly discounted. It may be thought that specifications that achieve R^2 values not too far from $R^2_{\max}$ would produce much narrower extreme bounds for β_0. It is this possibility that we consider in this paper. Suppose that $R^2_{\max}$ is found from the full model and $R^2_{\min}$ from the basic model (1). Consider model specifications achieving R^2 values equal to or greater than

$$R^2_\delta = (1-\delta)R^2_{\max} + \delta R^2_{\min},$$

where $0 < \delta < 1$.[1] For δ small these may be considered as being 'reasonable'

[1] See also Leamer (1981), in which similar ideas and results are obtained when constraining ridge estimates to achieve a given level of significance.

specifications as they are not far from the 'best' model in terms of goodness-of-fit, as measured by R^2. In the next section of the paper an equation for the values of the extreme bounds of β_0 is presented for any given δ. The proof is found in the appendix. A numerical example is presented in section 3, concerning the modelling of velocity and using time-series models. It is found that quite wide extreme bounds can occur using δ values as low as 0.1 or 0.2, relative to the extreme bounds found from the full set of possible specifications. This result strengthens Leamer's arguments about the difficulties that can arise when interpreting particular coefficients. Some further considerations are presented in the final section.

2. The model and results

The model being considered is

$$y = X\beta + \varepsilon, \tag{3}$$

where y is the vector of observations on the variable y and X is the matrix of observations on a vector of explanatory variables. At this stage, no distinction is being made between time-series or cross-section situations. It will be assumed that ε is $\mathrm{N}(0, \sigma^2\Omega)$ where the covariance matrix $\sigma^2\Omega$ is assumed known for the time being. The object of primary interest is the 'focus' coefficient

$$\beta_0 = \psi'\beta, \tag{4}$$

so that β_0 can be any individual coefficient or a weighted linear combination of coefficients. There will be a set of prior linear constraints,

$$C\beta = c. \tag{5}$$

We assume that C is nonredundant, i.e., has full row rank. It is convenient to use this general form for the constraints, but if the two types of variables X_{F}, X_{D} are considered, as in the first section, then the coefficients on X_{F} are free of restrictions (hence the notation β_{F}) and the coefficients on X_{D} are 'doubtful' in that they may or may not appear in any particular specification. The restrictions would then be $C = (0, I)$ with appropriate sizes of the zero and unit vectors, and $c = 0$. In a particular specification any linear combination of these restrictions can be used, so the objective is to study the range of estimators for β_0 when imposing linear constraints of the form

$$M(C\beta - c) = 0, \tag{6}$$

for some matrix M, which is assumed without loss of generality to be of full row rank. If no restrictions are placed on M, one gets the extreme bounds suggested by Leamer (1983). The following notation is used:

The generalized least-squares (GLS) estimates of β using the full model (3) with no exclusions is

$$b = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y,$$

which gives the estimate of β_0, $b_0 = \psi' b$.

Let

$$D = \sigma^2(X'\Omega^{-1}X)^{-1}, \qquad A = CDC',$$

and let $A^{-\frac{1}{2}}$ be a symmetric square root of A^{-1}.

For a given M in (6) define

$$W = A^{\frac{1}{2}}M'.$$

Two important vectors are

$$u = A^{-\frac{1}{2}}CD\psi \tag{7}$$

and

$$v = A^{-\frac{1}{2}}(Cb - c). \tag{8}$$

The Euclidean norm of u is $\|u\| = (u'u)^{\frac{1}{2}}$. It is convenient to define an angle $\theta \in [0, \pi/2]$ by

$$\cos 2\theta = \cos(u, v) \equiv \frac{u'v}{\|u\|\|v\|}, \tag{9}$$

and $\cos 2\theta = \cos(u, v) = 0$ if $u = 0$ or $v = 0$. The GLSE $\hat{\beta}$ of β_0 under the restriction (6) is

$$\hat{\beta}_0 = b_0 - u'W(W'W)^{-1}W'v.$$

Breusch (1985) proved:

Theorem 1. The extreme values of $\hat{\beta}_0$ over all choices of (full row rank matrices) M are

$$b_0 - \tfrac{1}{2}(\cos 2\theta \pm 1)\|u\|\|v\|,$$

i.e.,

$$b_0 - \cos^2\theta\|u\|\|v\| \quad \text{to} \quad b_0 + \sin^2\theta\|u\|\|v\|.$$

The bounds can be attained for some M.

Suppose now that a value for δ is chosen, with $0 \le \delta \le 1$, and models are considered having R^2 values[2] greater or equal to

$$R_\delta^2 = (1-\delta) R_{\max}^2 + \delta R_{\min}^2.$$

The upper and lower bounds on β_0 will be given by

$$b_0 - \phi_L \|u\| \|v\| \quad \text{and} \quad b_0 + \phi_U \|u\| \|v\|,$$

where ϕ_L and ϕ_U are always positive and depend on the chosen δ. The precise formula is given in:

Theorem 2. Assume that the row dimension of C is at least 2. Define an angle $\lambda \in [0, \pi/2]$ *such that*

$$\sin^2 \lambda = \delta.$$

(i) *If* $\lambda \le \theta$, *then*

$$\phi_U = \sin^2 \theta - \sin^2(\theta - \lambda),$$

and if $\lambda \ge \theta$, *then*

$$\phi_U = \sin^2 \theta.$$

(ii) *If* $\lambda \le \pi/2 - \theta$, *then*

$$\phi_L = \cos^2 \theta - \cos^2(\theta + \lambda),$$

and if $\lambda \ge \pi/2 - \theta$, *then*

$$\phi_L = \cos^2 \theta.$$

The proof is in the appendix.

For given δ, and thus λ, the extreme bounds can be found directly from the two regressions,[3] the basic regression involving just the free variables X_F,

[2]Alternatively, restrict M to be of rank m, with $0 < m < \text{rank}(C)$. Consider all models where the F-statistic for testing the set of linear constraints $M(C\beta - c)$ is less or equal to

$$F_{\delta, m} = \delta F_{m, \max} + (1-\delta) F_{m, \min},$$

where $F_{m, \max}$ is the maximum and $F_{m, \min}$ is the minimum among the achievable F-statistics when there are m restrictions. Again, Theorem 2 gives the correct bounds.

Observe that the F-statistic is essentially R^2 except for the number of regressors. Furthermore, any R^2 can be achieved for any given number m, $0 < m < \text{rank}(C)$, of restrictions: for the maximal R^2, simply include the regressor that arises as the linear combination of regressors with coefficients taken from the unrestricted regression. For the minimal R^2, choose regressors orthogonal to that regressor. Therefore fixing the number m is no restriction.

[3]Observe that the bounds do not depend on σ^2. Thus one might therefore set $\sigma^2 = 1$ to simplify the calculation.

which gives $\hat{b}_0$ as the estimate β_0 and also provides $R^2_{\min}$, and the complete regression in which all variables enter, which gives b as the estimate of β and b_0 as the estimate of β_0 and $R^2_{\max}$. It follows that

$$u'v = b_0 - \hat{b}_0,$$

$$\|v\| = (Cb - c')A^{-1}(Cb - c),$$

and

$$\|u\| = \mathrm{var}(b_0) - \mathrm{var}(\hat{b}_0).$$

From these quantities θ is determined as $\cos 2\theta = \cos(u, v)$. It may be suggested that it is good economic practice to report the values of $u'v$, $\|v\|$, $\|u\|$ as well as the extreme bounds on β_0 for various values of δ, say 0.1 and 0.05.

3. The effect of interest rates and inflation on the velocity of money

We want to analyze the effect of interest rates and inflation on the increase in the velocity of money. To this end, we consider two models in quarterly data and homoskedastic errors with unknown variance.

We used the following list of variables:

veloc = velocity of money, computed as GNP/M1,
dveloc = first differences in the velocity of money,
tbill = three month treasury-bill rate,
infla = inflation, computed from the consumer price index CPI,
gnp = gross national product,
mbvariab = variability of money,[4] computed as standard deviation within one year of the growth rate of the monetary base from its global mean and trend and 12 own lags, using monthly data. We multiply that number by 10,000 for numerical reasons.

GNP, CPI, M1, and monetary base are seasonally adjusted. We use first differences in the velocity of money in our regression.[5] The time index t counts quarters.

[4] It has been argued that the recent decline in the velocity of money was caused by changes in the variability of money supply [see Friedman (1984) and Hall and Noble (1987)].

[5] It has been argued that the velocity of money follows a random walk, i.e., that there are unit roots in the corresponding regression equation in levels. Then, first differencing the velocity series is a reasonable procedure. But also in the level model the inference drawn from OLS (instead of using unit-roots distribution theory) is valid, if the linear combination of regressors that we look at for our focus coefficient doesn't lie in the eigenspace of the unit root in the joint VAR [see Sims, Stock, and Watson (1986)]. Another way to justify using levels is the Bayesian point of view, in which the posterior distribution of the coefficient given the data is still almost normal even in the presence of unit roots [see Sims (1987)]. In fact, the results don't change much using levels instead of first differences except that the range of R^2 becomes much smaller. We chose a model in first differences because the results are more instructive and not because we believe the level model to yield incorrect results.

Model I

(a) *Unrestricted* ('equation 1')

$$dveloc(t) = \alpha + \beta_{0,0}t + \sum_{l=1}^{5} \beta_{l,1}dveloc(t-l) + \sum_{l=0}^{6} \beta_{l,2}tbill(t-l)$$

$$+ \sum_{l=0}^{5} \beta_{l,3}infla(t-l) + \sum_{l=0}^{6} \beta_{l,4}gnp(t-l)$$

$$+ \sum_{l=0}^{6} \beta_{l,5}mbvariab(t-l) + \varepsilon_t.$$

(b) *Restricted* ('equation 2')

$$dveloc(t) = \alpha + \beta_{0,0}t + \sum_{l=1}^{2} \beta_{l,1}dveloc(t-l) + \sum_{l=0}^{1} \beta_{l,2}tbill(t-l) + \varepsilon_t.$$

Model II

(a) *Unrestricted* ('equation 1'): Same as in model I

(b) *Restricted* ('equation 2')

$$dveloc(t) = \alpha - \beta_{0,0}t + \sum_{l=1}^{2} \beta_{l,1}dveloc(t-l) + \sum_{l=0}^{2} \beta_{l,2}tbill(t-l)$$

$$+ \sum_{l=0}^{2} \beta_{l,3}infla(t-l) + \sum_{l=0}^{1} \beta_{l,5}mbvariab(t-l) + \varepsilon_t.$$

Ordinary least squares were used to estimate the models.

Table 1 highlights a few results [the numbers for $\|u\|$, $\|v\|$, and $\cos(u, v)$ are calculated for $\sigma^2 = 1$, see remarks above]. These results contain several interesting aspects: The range of possible estimates for the trend variable is bigger than the two coefficient estimates in the restricted and the unrestricted model suggest. This coincides with the uncertainty about the trend coefficient as indicated by the t-statistic [see McAleer et al. (1985)].

Table 1

		Model I			
Focus coeff.		$\delta^2 = 1.0$	$\delta^2 = 0.1$	$\delta^2 = 0.05$	$\delta^2 = 0.0$
$\beta_{1,2}$(tbill – first lag)					
	Upper	0.08110	0.07342	0.06806	0.05154
	Lower	−0.00540	0.02420	0.03230	0.05154
	($\|u\| = 0.16853$, $\|v\| = 0.51325$, $\cos(u, v) = 0.31675$)				
$\beta_{0,0}$(trend)					
	Upper	0.00300	0.00161	0.00107	−0.00034
	Lower	−0.00342	−0.00224	−0.00172	−0.00034
	($\|u\| = 0.01251$, $\|v\| = 0.51325$, $\cos(u, v) = -0.0392$)				
		Model II			
Focus coeff.		$\delta^2 = 1.0$	$\delta^2 = 0.1$	$\delta^2 = 0.05$	$\delta^2 = 0.0$
$\beta_{1,2}$(tbill – first lag)					
	Upper	0.07635	0.07044	0.06591	0.05155
	Lower	−0.00108	0.02709	0.03441	0.05155
	($\|u\| = 0.15999$, $\|v\| = 0.48395$, $\cos(u, v) = 0.35931$)				
$\beta_{2,2}$(tbill – second lag)					
	Upper	0.00725	−0.02404	−0.03436	−0.06019
	Lower	−0.10847	−0.09251	−0.08411	−0.06019
	($\|u\| = 0.23911$, $\|v\| = 0.48395$, $\cos(u, v) = -0.16570$)				
$\sum_{l=0}^{6} \beta_{l,2}$(effect of permanent increase in interest rates)					
	Upper	0.01682	0.01288	0.01048	0.00334
	Lower	−0.01911	−0.00799	−0.00469	0.00334
	($\|u\| = 0.23911$, $\|v\| = 0.48395$, $\cos(u, v) = -0.16570$)				
$\beta_{1,3}$(inflation – first lag)					
	Upper	0.01156	0.00918	0.00795	0.00448
	Lower	−0.00513	−0.00072	0.00076	0.00448
	($\|u\| = 0.03449$, $\|v\| = 0.48395$, $\cos(u, v) = 0.15132$)				

The shape of the range of coefficient estimates for the first lag of *tbill* changes little in either model. In both cases it has a positive coefficient as long as we stay in the top 20% of R^2, say. Similarly, the second lag has a negative coefficient. Notice that the unrestricted EBA doesn't allow here for that conclusion (since coefficients of the opposite sign in these cases are included here when R^2 is not restricted). It is of course subject to debate whether it is reasonable to look at a fraction of the range of possible R^2. The point is that we have to make the unrestricted model pretty much as bad as possible within the admissible range to arrive at coefficients of the opposite sign in either case.

It is now up to the judgement of the individual researcher to decide whether (s)he wants to rule out these coefficients as unreasonable (because of the 'bad' R^2) or not. This type of sensitivity analysis, with the basic facts exposed, is of course the whole point of this approach.

We also see that not much can be said about the permanent effects of, e.g., a permanent rise in the nominal interest rate. Different models allow for different conclusions within the top 20% range of R^2 and that might be all that can be said. The same applies to money-base variability which we did not find to have a clear effect on velocity. Looking at a plot of real interest rates (computed as $tbill(t) - inflation(t)$, where contemporaneous inflation is used as a crude substitute for the inflation expectation of agents in our economy), it seems that the recent change in the behavior of the velocity of money coincides with the shift of real interest rates from negative values to positive values.

An analysis as shown here helps to understand better the possibilities for the outcomes from different models and the type of restrictions we impose when passing from a large 'benchmark' model to a smaller model.

4. Conclusions

Our result enables the researcher with a 'continuum' of choices between classical econometrics and Leamer's 'extreme' EBA: if one only wants the maximal R^2, the theorem will give the coefficient of the classical analysis. If one allows for any R^2, the theorem gives the extreme bounds as, e.g., given by Breusch's theorem (which is contained in our theorem as a special result). It seems reasonable, as explained above, to give the extreme bounds of the coefficient of interest subject to restricting R^2 to be in the top 5% or top 10% of the range of possible R^2: using ordinary-regression output, this can be done using the formulas of the theorem. Of course, one would like extensions of our result: How can we deal with a vector of coefficients of interest? Is there a similar version of the theorem that controls for the Durbin–Watson statistic? How can we incorporate uncertainty about the covariance matrix Ω in our model? Are there similar results for other inference procedures (such as probit models) or in the presence of nonnormal distributions (as in the unit-roots case, for example)? How can one include coefficient-uncertainty arising from any of the specific models included in our range? Can bounds be found if positivity constraints are imposed on certain coefficients? Are there interesting asymptotic results?

The theorem above allows us to get a feel of how much actually changes, if we proceed from a general 'benchmark' model to a smaller, restricted model. The procedure described above is intended to append current practice in that way and not to replace it, and can thus provide useful insights about the data.

Appendix 1: Proof of the theorem

Let $k \geq 2$ be the number of possible restrictions, i.e., the row dimension of C. Note that we get any full column rank matrix W as M ranges over all full row rank matrices and vice versa. It therefore suffices to restrict attention to full column rank matrices W.

Theorem 2 is correct if $\|u\| = 0$ or $\|v\| = 0$: this follows directly from the formula for $\hat{\beta}_0$. Hence assume $\|u\| \neq 0$ and $\|v\| \neq 0$.

Define functions

$$\varphi(W) = u'W(W'W)^{-1}W'v/(\|u\|\|v\|),$$

$$\gamma(W) = v'W(W'W)^{-1}W'v/(\|v\|\|v\|).$$

Observe that $0 \leq \gamma(W) \leq 1$ and that these bounds are sharp, since $W(W'W)^{-1}W'$ is a matrix that maps any k-dimensional vector in an orthogonal way on the range of W.

Fix $\delta \in [0,1]$ and choose W. A little calculation shows that the GLSE $\hat{\beta}_0$ of β_0 [under the restriction $W'A^{-\frac{1}{2}}(C\beta - c) = 0$] and the corresponding correlation coefficient R^2 are given by

$$\hat{\beta}_0 = b_0 - \varphi(W)\|u\|\|v\|,$$

$$R^2 = 1 - \frac{e'e + \gamma(W)\|v\|^2\sigma^2}{(y-\bar{y})'(y-\bar{y})},$$

where e is the vector of residuals under the unrestricted model and $\bar{y}$ is the mean of y.[6] Consequently, $R^2 \geq R^2_\delta$ is equivalent to $\gamma(W) \leq \delta$.

If $\{u, v\}$ is linear dependent, then there are two cases: $\cos(u,v) = 1$ or $\cos(u,v) = -1$. Suppose that $\gamma(W) \leq \delta$. In the first case,

$$\phi_{\mathrm{U}} = 0 \leq \varphi(W) = \gamma(W) \leq \delta = 1 - \cos^2\lambda = \phi_{\mathrm{L}},$$

and in the second case,

$$\phi_{\mathrm{L}} = 0 \leq -\varphi(W) = \gamma(W) \leq \delta = 1 - \cos^2\lambda = \phi_{\mathrm{U}}.$$

These bounds can be achieved: find a vector $x \neq 0$ orthogonal to v. Such a vector exists since $k \geq 2$. Let $W(t) = tx + (1-t)v$. Observe that $\varphi(W(0)) = \cos(u,v)$, $\varphi(W(1)) = 0$, and that φ is continuous. Hence, for any $\phi \in [0,1]$,

[6] We have, of course, $\bar{y} = 0$ if the model is already in deviation form.

there is some t so that

$$\varphi(W(t))\cos(u,v)=\gamma(W(t))=\phi.$$

This proves Theorem 2 for the case where $\{u,v\}$ is linear dependent.

Assume now that $\{u,v\}$ is linear independent. The proof involves two steps. First, we show that any W can do only 'worse' than some matrix $x\neq 0$ of the form $x=au+bv$, i.e., at the same R^2, $\varphi(W)$ will be restricted to the range given by the $\varphi(x)$ for some a and b. Secondly, we show that we get the bounds mentioned in the theorem within this subset of matrices that can be written as linear combinations of u and v, thus proving the theorem. To simplify the calculations, renormalize u and v so that $\|u\|=\|v\|=1$ (we use the same notation for the renormalized vectors).

Claim. For all W, there is a linear combination $x\neq 0$ of u and v and a real number η with $0\leq\eta\leq 1$ such that $\gamma(W)=\gamma(x)$ and $\varphi(W)=\eta\varphi(x)$.

Proof of the claim. Fix W. Let P be the orthogonal projection on the plane spanned by u and v. Let $y=W(W'W)^{-1}W'v$ and $z=Py$. Then $\varphi(W)=u'z$, and similarly $\gamma(W)=v'z$. If $z=0$, we are done: choose a linear combination x of u and v which is orthogonal to v and choose $\eta=0$. Hence, assume $z\neq 0$. Observe that $z'z\leq y'y=y'v=z'v$ by the definition of P. Let q be orthogonal to v in the plane spanned by u and v, of unit length, and so that $q'u\geq 0$. Set $x=(z'v)v+\alpha q$, where α is the solution to the equation $\alpha^2=(z'v)(1-z'v)$ that satisfies $\alpha(u'z)\geq 0$. Check that $z'v=x'v=x'x$ and thus $x=x(x'x)^{-1}x'v$. Furthermore, $\|x\|^2=z'v\geq\|z\|^2$. Therefore we can write z as $z=(z'v)v+\tau\alpha q$, where $|\tau|\leq 1$. We find that $\gamma(x)=v'x=v'z=\gamma(W)$ and (for the case $u'z\neq 0$, otherwise the claim is now trivial) $\varphi(x)=u'x=u'z+(1-\tau)\alpha u'q=\varphi(W)/\eta$, where $1/\eta=1+(1-\tau)\alpha(u'q)/(u'z)\geq 1$. This finishes the first part of the proof.

Attention is now turned to the second part of the proof:

Since $\varphi(W)=\varphi(sW)$ and $\gamma(W)=\gamma(sW)$ for any nonzero scalar s, it is enough to consider only matrices of the form

$$W_\rho=\sin(\rho)v+\cos(\rho)q,$$

for $\rho\in[-\pi/2,\pi/2]$ (recall from above, that q was defined to be orthogonal to v in the u–v plane, of unit length, and so that $u'q\geq 0$). Calculate

$$\gamma(W_\rho)=\sin^2(\rho).$$

This implies that for the matrices W_ρ, $R^2 \geq R_\delta^2$ is equivalent to $|\rho| \leq \lambda$. Furthermore, we find

$$u'W_\rho = \sin(\rho)\cos(2\theta) + \cos(\rho)u'q = \sin(\rho + 2\theta),$$

since $u = (u'q)q + \cos(2\theta)v$, $\theta \in [0, \pi/2]$, and $u'q \geq 0$. With the help of the usual theorems about cos and sin, we thus have

$$-\varphi(W_\rho) = -\sin(\rho)\sin(\rho + 2\theta) = -\left(\cos^2(\theta) - \cos^2(\theta + \rho)\right)$$

$$= \sin^2(\theta) - \sin^2(\theta + \rho).$$

Observe now, that

$$\cos^2(\theta - \rho) \geq \max\{\cos(\theta + \rho); 0\}^2,$$

since $\theta \in [0, \pi/2]$. Hence,

$$-\varphi(W_\rho) \geq -\phi_{\mathrm{L}} \quad \text{if} \quad |\rho| \leq \lambda,$$

where ϕ_{L} is the bound stated in the theorem. The bound is achieved at $\rho = \min\{\lambda; \pi/2 - \theta\}$. Likewise,

$$-\varphi(W_\rho) \leq \phi_{\mathrm{U}} \quad \text{if} \quad |\rho| \leq \lambda,$$

and the bound is achieved for $\rho = -\min\{\lambda; \theta\}$. This completes the proof of the theorem.

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