Econometrica Supplementary Material

SUPPLEMENT TO "PARTIAL IDENTIFICATION IN TRIANGULAR SYSTEMS OF EQUATIONS WITH BINARY DEPENDENT VARIABLES": APPENDIX

(*Econometrica*, Vol. 79, No. 3, May 2011, 949–955)

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PROOF OF LEMMA 2.1: First recall the simplifications following from Assumptions 2.1 and 2.2 noted at the beginning of Section 2. Next, note from equation (1) and Assumption 2.1 that

$$\Pr\{D = 1, Y = 1 | X = x', P = p\} = \Pr\{\varepsilon_2 \le p, \varepsilon_1 \le \nu_1(1, x')\}$$

and

$$\Pr\{D = 1, Y = 1 | X = x', P = p'\} = \Pr\{\varepsilon_2 \le p', \varepsilon_1 \le \nu_1(1, x')\}.$$

Thus, for $p > p'$,

$$\Pr\{D = 1, Y = 1 | X = x', P = p\} - \Pr\{D = 1, Y = 1 | X = x', P = p'\}$$

is equal to

$$\Pr\{p' < \varepsilon_2 \le p, \varepsilon_1 \le \nu_1(1, x')\}.$$

It follows similarly that

$$\Pr\{D = 0, Y = 1 | X = x, P = p'\} = \Pr\{\varepsilon_2 > p', \varepsilon_1 \le \nu_1(0, x)\}$$

and

$$\Pr\{D = 0, Y = 1 | X = x, P = p\} = \Pr\{\varepsilon_2 > p, \varepsilon_1 \le \nu_1(0, x)\}.$$

Therefore,

$$\Pr\{D = 0, Y = 1 | X = x, P = p'\} - \Pr\{D = 0, Y = 1 | X = x, P = p\}$$

is equal to

$$\Pr\{p' < \varepsilon_2 \le p, \varepsilon_1 \le \nu_1(0, x)\}.$$

Hence,

$$h(x, x', p, p') = \begin{cases} \Pr\{p' < \varepsilon_2 \le p, \nu_1(0, x) < \varepsilon_1 \le \nu_1(1, x')\} \\ \quad \text{if } \nu_1(1, x') > \nu_1(0, x), \\ 0 \\ \quad \text{if } \nu_1(1, x') = \nu_1(0, x), \\ -\Pr\{p' < \varepsilon_2 \le p, \nu_1(1, x') < \varepsilon_1 \le \nu_1(0, x)\} \\ \quad \text{if } \nu_1(1, x') < \nu_1(0, x). \end{cases}$$

 DOI: 10.3982/ECTA9082

The desired conclusion now follows immediately from Assumption 2.2. *Q.E.D.*

PROOF OF THEOREM 2.1: Consider part (i) of the theorem. We derive bounds on $G_1(0,x) = \Pr\{Y_0 = 1|X = x\}$; the bounds on $G_1(1,x)$ and on $\Delta G_1(x)$ follow from parallel arguments.

Note that

$$\Pr\{Y_0 = 1|X = x, P = p\} = \Pr\{D = 0, Y_0 = 1|X = x, P = p\} + \Pr\{D = 1, Y_0 = 1|X = x, P = p\}.$$

By Lemma 2.1, equation (1), and Assumption 2.1,

$$\Pr\{D = 1, Y_0 = 1|X = x, P = p\} \le \Pr\{D = 1, Y = 1|X = x', P = p\}$$

for all $x' \in \mathbf{X}_{0+}(x)$ and

$$\Pr\{D = 1, Y_0 = 1|X = x, P = p\} \ge \Pr\{D = 1, Y = 1|X = x', P = p\}$$

for all $x' \in \mathbf{X}_{0-}(x)$. Thus, $\Pr\{Y_0 = 1|X = x, P = p\}$ is bounded from below by

$$\Pr\{D = 0, Y = 1|X = x, P = p\} + \sup_{x' \in \mathbf{X}_{0-}(x)} \Pr\{D = 1, Y = 1|X = x', P = p\}$$

and from above by

$$\Pr\{D = 0, Y = 1|X = x, P = p\} + p \inf_{x' \in \mathbf{X}_{0+}(x)} \Pr\{Y = 1|D = 1, X = x', P = p\},$$

where all supremums and infimums are only taken over regions where all conditional probabilities are well defined, and with the convention that the supremum over the empty set is 0 and the infimum over the empty set is 1. The stated result now follows by noting that equation (1) and Assumption 2.1 imply that $\Pr\{Y_0 = 1|X = x\} = \Pr\{Y_0 = 1|X = x, P = p\}$.

Consider part (ii) of the theorem. We prove the result for the term $L_0(x)$; the result for the other terms follows from parallel arguments.

Suppose $\mathrm{supp}(P)$ is not a singleton, for otherwise there is nothing to prove. Since $\mathrm{supp}(X, P) = \mathrm{supp}(X) \times \mathrm{supp}(P)$, $h(x, x', p, p')$ is well defined for some $p < p'$ with $(p, p') \in \mathrm{supp}(P)^2$ and any $(x, x') \in \mathrm{supp}(X)^2$. Hence, by Lemma 2.1, we have that

$$\mathbf{X}_{0-}(x) = \{x' : \nu_1(1, x') \le \nu_1(0, x)\}. \tag{4}$$

It follows from Assumptions 2.3 and 2.4 that $\mathbf{X}_{0-}(x)$ is compact. Hence, by Assumption 2.4, there exists $x_0^l(x) \in \mathbf{X}_{0-}(x)$ such that

$$\nu_1(1, x_0^l(x)) = \sup_{x' \in \mathbf{X}_{0-}(x)} \nu_1(1, x').$$

From equation (1), we therefore have for any $p \in \mathrm{supp}(P)$ that

$$\begin{aligned}\sup_{x' \in \mathbf{X}_{0-}(x)} & \Pr\{D = 1, Y = 1 | X = x', P = p\} \\ &= \Pr\{D = 1, Y = 1 | X = x_0^l(x), P = p\},\end{aligned}$$

from which it follows that

$$\begin{aligned}L_0(x) = \sup_p \big\{ & \Pr\{D = 0, Y = 1 | X = x, P = p\} \\ & + \Pr\{D = 1, Y = 1 | X = x_0^l(x), P = p\}\big\}.\end{aligned}$$

To complete the argument, note for any $p > p'$ that

$$\begin{aligned}&\big(\Pr\{D = 0, Y = 1 | X = x, P = p\} \\ &\qquad + \Pr\{D = 1, Y = 1 | X = x_0^l(x), P = p\}\big) \\ &\qquad - \big(\Pr\{D = 0, Y = 1 | X = x, P = p'\} \\ &\qquad + \Pr\{D = 1, Y = 1 | X = x_0^l(x), P = p'\}\big) \\ &\quad = \Pr\big\{\varepsilon_1 \le \nu_1(1, x_0^l(x)), p' < \varepsilon_2 \le p\big\} \\ &\qquad - \Pr\{\varepsilon_1 \le \nu_1(0, x), p' < \varepsilon_2 \le p\} \\ &\quad \le 0,\end{aligned}$$

where the final inequality follows from the fact that $x_0^l(x) \in \mathbf{X}_{0-}(x)$ and (4).

Finally, consider part (iii) of the theorem. Before proceeding, we introduce some notation. Let $(\varepsilon_1^*, \varepsilon_2^*)$ denote a random vector with $(\varepsilon_1^*, \varepsilon_2^*) \perp\!\!\!\perp (X, Z)$ and with $(\varepsilon_1^*, \varepsilon_2^*)$ having density $f_{1,2}^*$ with respect to Lebesgue measure on $\mathbf{R}^2$. Let f_2^* denote the corresponding marginal density of ε_2^* and let $f_{1|2}^*$ denote the corresponding density of ε_1^* conditional on ε_2^*. Let $f_{1,2}$, $f_{1|2}$, and f_2 denote the corresponding density functions for $(\varepsilon_1, \varepsilon_2)$. We will also make use of $F_{1,2}$, the cumulative distribution function (c.d.f.) for $(\varepsilon_1, \varepsilon_2)$, and $F_{1,-2}$, the c.d.f. for $(\varepsilon_1, -\varepsilon_2)$.

To show that our bounds on $G_1(0, x)$, $G_1(1, x)$, and $G_1(1, x) - G_1(0, x)$ are sharp, it suffices to show that for any $x \in \mathrm{supp}(X)$ and $(s_0, s_1) \in [L_0(x), U_0(x)] \times [L_1(x), U_1(x)]$, there exists a density function $f_{1,2}^*$ such that the following claims hold:

(A) $f^*_{1,2}$ is strictly positive on $\mathbf{R}^2$.

(B) the proposed model is consistent with the observed data, that is,

(i) $\Pr\{D=1|X=\tilde{x}, P=p\} = \Pr\{\varepsilon^*_2 \le p\}$,

(ii) $\Pr\{Y=1|D=1, X=\tilde{x}, P=p\} = \Pr\{\varepsilon^*_1 \le \nu_1(1,\tilde{x})|\varepsilon^*_2 \le p\}$,

(iii) $\Pr\{Y=1|D=0, X=\tilde{x}, P=p\} = \Pr\{\varepsilon^*_1 \le \nu_1(0,\tilde{x})|\varepsilon^*_2 > p\}$

for all $(\tilde{x}, p) \in \text{supp}(X, P)$.

(C) The proposed model is consistent with the specified values of $G_1(0, x)$ and $G_1(1, x)$, that is,

(i) $\Pr\{\varepsilon^*_1 \le \nu_1(0, x)\} = s_0$,

(ii) $\Pr\{\varepsilon^*_1 \le \nu_1(1, x)\} = s_1$.

Let $x \in \text{supp}(X)$ and $(s_0, s_1) \in [L_0(x), U_0(x)] \times [L_1(x), U_1(x)]$ be given. We prove the result for the case where $\mathbf{X}_{d-}(x) \neq \emptyset$, $\mathbf{X}_{d+}(x) \neq \emptyset$, and $\mathbf{X}_{d-}(x) \cap \mathbf{X}_{d+}(x) = \emptyset$ for $d \in \{0, 1\}$; the result in the other cases follows from analogous arguments. Note that by arguing as in Remark 2.2, this implies in particular that $L_d(x) < U_d(x)$ for $d \in \{0, 1\}$. For brevity, we also only consider $(s_0, s_1) \in (L_0(x), U_0(x)) \times (L_1(x), U_1(x))$; the case where s_d equals $L_d(x)$ or $U_d(x)$ for some $d \in \{0, 1\}$ follows from a straightforward modification of the argument below.

Recall that $h(x, x', p, p')$ is well defined for some $p < p'$ with $(p, p') \in \text{supp}(P)^2$ and any $(x, x') \in \text{supp}(X)^2$ because $\text{supp}(X, P) = \text{supp}(X) \times \text{supp}(P)$. Arguing as in the proof of part (ii) of the theorem, we have that

$$
\begin{aligned}
(5) \qquad L_0(x) &= \Pr\{D=0, Y=1|X=x, P=\underline{p}\} \\
&\quad + \Pr\{D=1, Y=1|X=x^l_0(x), P=\underline{p}\}, \\
U_0(x) &= \Pr\{D=0, Y=1|X=x, P=\underline{p}\} \\
&\quad + \Pr\{D=1, Y=1|X=x^u_0(x), P=\underline{p}\}, \\
L_1(x) &= \Pr\{D=1, Y=1|X=x, P=\overline{p}\} \\
&\quad + \Pr\{D=0, Y=1|X=x^l_1(x), P=\overline{p}\}, \\
U_1(x) &= \Pr\{D=1, Y=1|X=x, P=\overline{p}\} \\
&\quad + \Pr\{D=0, Y=1|X=x^u_1(x), P=\overline{p}\},
\end{aligned}
$$

where $x^l_d(x)$ and $x^u_d(x)$ for $d \in \{0, 1\}$ denote evaluation points such that

$$
\begin{aligned}
&\Pr\{D=1, Y=1|X=x^l_0(x), P=\underline{p}\} \\
&\quad = \sup_{x' \in \mathbf{X}_{0-}(x)} \Pr\{D=1, Y=1|X=x', P=\underline{p}\}, \\
&\Pr\{D=1, Y=1|X=x^u_0(x), P=\underline{p}\} \\
&\quad = \inf_{x' \in \mathbf{X}_{0+}(x)} \Pr\{D=1, Y=1|X=x', P=\underline{p}\},
\end{aligned}
$$

$$\Pr\{D=0, Y=1|X=x_1^l(x), P=\overline{p}\} = \sup_{x'\in \mathbf{X}_{1+}(x)} \Pr\{D=0, Y=1|X=x', P=\overline{p}\},$$
$$\Pr\{D=0, Y=1|X=x_1^u(x), P=\overline{p}\} = \inf_{x'\in \mathbf{X}_{1-}(x)} \Pr\{D=0, Y=1|X=x', P=\overline{p}\}.$$

Let

$$s_0^* = s_0 - \Pr\{D=0, Y=1|X=x, P=\underline{p}\},$$
$$s_1^* = s_1 - \Pr\{D=1, Y=1|X=x, P=\overline{p}\}.$$

Using equation (5) and the fact that $s_d \in (L_d(x), U_d(x))$ for $d \in \{0,1\}$, we have that

$$s_0^* \in \big(F_{1,2}(\nu_1(1, x_0^l(x)), \underline{p}), F_{1,2}(\nu_1(1, x_0^u(x)), \underline{p})\big),$$
$$s_1^* \in \big(F_{1,-2}(\nu_1(0, x_1^l(x)), -\overline{p}), F_{1,-2}(\nu_1(0, x_1^u(x)), -\overline{p})\big). \tag{6}$$

These intervals are nonempty because $L_d(x) < U_d(x)$ for $d \in \{0,1\}$. It follows by Lemma 2.1 that

$$\nu_1(d, x_{1-d}^l(x)) < \nu_1(1-d, x) < \nu_1(d, x_{1-d}^u(x)) \tag{7}$$

for $d \in \{0,1\}$, where the strict inequalities follow from our assumption that $\mathbf{X}_{d-}(x) \cap \mathbf{X}_{d+}(x) = \emptyset$ for $d \in \{0,1\}$. Furthermore, by the construction of $x_d^l(x)$ and $x_d^u(x)$ for $d \in \{0,1\}$, it must be the case for $d \in \{0,1\}$ and $\tilde{x} \in \mathrm{supp}(X)$ that

$$\nu_1(d, \tilde{x}) \notin \big(\nu_1(d, x_{1-d}^l(x)), \nu_1(d, x_{1-d}^u(x))\big). \tag{8}$$

We now construct the proposed density $f_{1,2}^*$ as follows. Let $f_{1,2}^*(t_1,t_2) = f_{1|2}^*(t_1|t_2) f_2^*(t_2)$, where $f_2^*(t_2) = f_2(t_2) = I\{0 \le t_2 \le 1\}$ and

$$f_{1|2}^*(t_1|t_2) = \begin{cases} a(t_2) f_{1|2}(t_1|t_2) \\ \quad \text{if } \nu_1(1, x_0^l(x)) < t_1 < \nu_1(0,x) \text{ and } t_2 < \underline{p}, \\ b(t_2) f_{1|2}(t_1|t_2) \\ \quad \text{if } \nu_1(0,x) \le t_1 < \nu_1(1, x_0^u(x)) \text{ and } t_2 < \underline{p}, \\ c(t_2) f_{1|2}(t_1|t_2) \\ \quad \text{if } \nu_1(0, x_1^l(x)) \le t_1 < \nu_1(1,x) \text{ and } t_2 > \overline{p}, \\ d(t_2) f_{1|2}(t_1|t_2) \\ \quad \text{if } \nu_1(1,x) \le t_1 < \nu_1(0, x_1^u(x)) \text{ and } t_2 > \overline{p}, \\ f_{1|2}(t_1|t_2) \\ \quad \text{otherwise,} \end{cases}$$

with

$$
\begin{aligned}
a(t_2) &= \frac{\Pr\{\nu_1(1, x_0^l(x)) < \varepsilon_1 < \nu_1(1, x_0^u(x))|\varepsilon_2 = t_2\}}{\Pr\{\nu_1(1, x_0^l(x)) < \varepsilon_1 < \nu_1(0, x)|\varepsilon_2 = t_2\}} \\
&\quad \times \frac{s_0^* - F_{1,2}(\nu_1(1, x_0^l(x)), \underline{p})}{F_{1,2}(\nu_1(1, x_0^u(x)), \underline{p}) - F_{1,2}(\nu_1(1, x_0^l(x)), \underline{p})}, \\
b(t_2) &= \big(\Pr\{\nu_1(1, x_0^l(x)) < \varepsilon_1 < \nu_1(1, x_0^u(x))|\varepsilon_2 = t_2\} \\
&\quad - a(t_2)\Pr\{\nu_1(1, x_0^l(x)) < \varepsilon_1 < \nu_1(0, x)|\varepsilon_2 = t_2\}\big) \\
&\quad / \Pr\{\nu_1(0, x) < \varepsilon_1 < \nu_1(1, x_0^u(x))|\varepsilon_2 = t_2\}, \\
c(t_2) &= \frac{\Pr\{\nu_1(0, x_1^l(x)) < \varepsilon_1 < \nu_1(0, x_1^u(x))|\varepsilon_2 = t_2\}}{\Pr\{\nu_1(0, x_1^l(x)) < \varepsilon_1 < \nu_1(1, x)|\varepsilon_2 = t_2\}} \\
&\quad \times \frac{s_1^* - F_{1,-2}(\nu_1(0, x_1^l(x)), -\overline{p})}{F_{1,-2}(\nu_1(0, x_1^u(x)), -\overline{p}) - F_{1,-2}(\nu_1(0, x_1^l(x)), -\overline{p})}, \\
d(t_2) &= \big(\Pr\{\nu_1(0, x_1^l(x)) < \varepsilon_1 < \nu_1(0, x_1^u(x))|\varepsilon_2 = t_2\} \\
&\quad - c(t_2)\Pr\{\nu_1(0, x_1^l(x)) < \varepsilon_1 < \nu_1(1, x)|\varepsilon_2 = t_2\}\big) \\
&\quad / \Pr\{\nu_1(1, x) < \varepsilon_1 < \nu_1(0, x_0^u(x))|\varepsilon_2 = t_2\}.
\end{aligned}
$$

These quantities are well defined because of the fact that the intervals in (6) are nonempty, because of (7), and Assumption 2.2.

We now argue that $f_{1,2}^*$ satisfies claim (A), that is, that it is a strictly positive density on $\mathbf{R}^2$. For this purpose, it suffices to show that $f_{1|2}^*$ integrates to 1 and is strictly positive on $\mathbf{R}$. First consider whether $f_{1|2}^*$ integrates to 1. For $t_2 \in [\underline{p}, \overline{p}]$, $f_{1|2}^*(\cdot|t_2) = f_{1|2}(\cdot|t_2)$ and so the result follows immediately. For $t_2 < \underline{p}$,

$$
\begin{aligned}
&\int_{-\infty}^{\infty} f_{1|2}^*(t_1|t_2)\,dt_1 \\
&\quad = \int_{-\infty}^{\nu_1(1,x_0^l(x))} f_{1|2}(t_1|t_2)\,dt_1 + a(t_2)\int_{\nu_1(1,x_0^l(x))}^{\nu_1(0,x)} f_{1|2}(t_1|t_2)\,dt_1 \\
&\qquad + b(t_2)\int_{\nu_1(0,x)}^{\nu_1(1,x_0^u(x))} f_{1|2}(t_1|t_2)\,dt_1 + \int_{\nu_1(1,x_0^u(x))}^{\infty} f_{1|2}(t_1|t_2)\,dt_1 \\
&\quad = \Pr\big\{\varepsilon_1 \le \nu_1(1, x_0^l(x))|\varepsilon_2 = t_2\big\} \\
&\qquad + \Pr\big\{\nu_1(1, x_0^l(x)) < \varepsilon_1 < \nu_1(1, x_0^u(x))|\varepsilon_2 = t_2\big\} \\
&\qquad + \Pr\{\varepsilon_1 \ge \nu_1(1, x_0^u(x))|\varepsilon_2 = t_2\} \\
&\quad = 1.
\end{aligned}
$$

A similar argument shows that $\int f^*_{1|2}(t_1|t_2)\,dt_1 = 1$ for $t_2 > \overline{p}$.

Since $f_{1|2}$ is strictly positive on $\mathbf{R}$, to establish that $f^*_{1|2}$ is strictly positive on $\mathbf{R}$, it suffices to show that $a(t_2), b(t_2), c(t_2)$, and $d(t_2)$ are all strictly positive. Consider $a(t_2)$ and $b(t_2)$; the proof for $c(t_2)$ and $d(t_2)$ follows from similar arguments. From (6), we have that $s^*_0 > F_{1,2}(\nu_1(1, x^l_0(x)), \underline{p})$, which together with (7) and Assumption 2.2 implies that $a(t_2) > 0$. Similarly, from (6), we have that $s^*_0 < F_{1,2}(\nu_1(1, x^u_0(x)), \underline{p})$, which implies that

$$\frac{s^*_0 - F_{1,2}(\nu_1(1, x^l_0(x)), \underline{p})}{F_{1,2}(\nu_1(1, x^u_0(x)), \underline{p}) - F_{1,2}(\nu_1(1, x^l_0(x)), \underline{p})} < 1.$$

It therefore follows from (7) and Assumption 2.2 that

$$\begin{aligned}
&\Pr\bigl\{\nu_1(1, x^l_0(x)) < \varepsilon_1 < \nu_1(1, x^u_0(x))|\varepsilon_2 = t_2\bigr\} \\
&\qquad - a(t_2)\Pr\bigl\{\nu_1(1, x^l_0(x)) < \varepsilon_1 < \nu_1(0, x)|\varepsilon_2 = t_2\bigr\} \\
&\quad = \Pr\bigl\{\nu_1(1, x^l_0(x)) < \varepsilon_1 < \nu_1(1, x^u_0(x))|\varepsilon_2 = t_2\bigr\} \\
&\qquad \times \left(1 - \frac{s^*_0 - F_{1,2}(\nu_1(1, x^l_0(x)), \underline{p})}{F_{1,2}(\nu_1(1, x^u_0(x)), \underline{p}) - F_{1,2}(\nu_1(1, x^l_0(x)), \underline{p})}\right) \\
&\quad > 0,
\end{aligned}$$

so $b(t_2) > 0$.

We now argue that $f^*_{1,2}$ satisfies claim (B). Since $f^*_2 = f_2$, we have immediately that $\Pr\{\varepsilon^*_2 \le p\} = \Pr\{D = 1|X = \tilde{x}, P = p\}$ for all $(\tilde{x}, p) \in \operatorname{supp}(X, P)$. Consider $\Pr\{\varepsilon^*_1 \le \nu_1(1, \tilde{x})|\varepsilon^*_2 \le p\}$. From (8), we have that $\nu_1(1, \tilde{x}) \le \nu_1(1, x^l_0(x))$ or $\nu_1(1, \tilde{x}) \ge \nu_1(1, x^u_0(x))$ for any $\tilde{x} \in \operatorname{supp}(X)$. For $(\tilde{x}, p) \in \operatorname{supp}(X, P)$ such that $\nu_1(1, \tilde{x}) \le \nu_1(1, x^l_0(x))$, we have

$$\begin{aligned}
&\Pr\{\varepsilon^*_1 \le \nu_1(1, \tilde{x})|\varepsilon^*_2 \le p\} \\
&\quad = \frac{1}{p}\int_0^p \int_{-\infty}^{\nu_1(1,\tilde{x})} f^*_{1,2}(t_1, t_2)\,dt_1\,dt_2 \\
&\quad = \frac{1}{p}\int_0^p \int_{-\infty}^{\nu_1(1,\tilde{x})} f_{1,2}(t_1, t_2)\,dt_1\,dt_2 \\
&\quad = \Pr\{\varepsilon_1 \le \nu_1(1, \tilde{x})|\varepsilon_2 \le p\} = \Pr\{Y = 1|D = 1, X = \tilde{x}, P = p\}.
\end{aligned}$$

For $(\tilde{x}, p) \in \operatorname{supp}(X, P)$ such that $\nu_1(1, \tilde{x}) \ge \nu_1(1, x^u_0(x))$, we have

$$\begin{aligned}
&\Pr\{\varepsilon^*_1 \le \nu_1(1, \tilde{x})|\varepsilon^*_2 \le p\} \\
&\quad = \frac{1}{p}\int_0^p \int_{-\infty}^{\nu_1(1,\tilde{x})} f^*_{1,2}(t_1, t_2)\,dt_1\,dt_2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{p}\bigg\{\int_{\underline{p}}^{p}\int_{-\infty}^{\nu_1(1,\tilde{x})} f_{1,2}(t_1,t_2)\,dt_1\,dt_2 \\
&\quad + \int_0^{\underline{p}}\bigg[\int_{-\infty}^{\nu_1(1,x_0^l(x))} f_{1|2}(t_1|t_2)\,dt_1 + a(t_2)\int_{\nu_1(1,x_0^l(x))}^{\nu_1(0,x)} f_{1|2}(t_1|t_2)\,dt_1 \\
&\quad + b(t_2)\int_{\nu_1(0,x)}^{\nu_1(1,x_0^u(x))} f_{1|2}(t_1|t_2)\,dt_1 + \int_{\nu_1(1,x_0^u(x))}^{\nu_1(1,\tilde{x})} f_{1|2}(t_1|t_2)\,dt_1\bigg]\,dt_2\bigg\} \\
&= \frac{1}{p}\big\{\Pr\{\varepsilon_1 \le \nu_1(1,\tilde{x}),\ \underline{p} < \varepsilon_2 \le p\} + \Pr\{\varepsilon_1 \le \nu_1(1,\tilde{x}),\ \varepsilon_2 \le \underline{p}\}\big\} \\
&= \Pr\{\varepsilon_1 \le \nu_1(1,\tilde{x}) | \varepsilon_2 \le p\} = \Pr\{Y=1|D=1, X=\tilde{x}, P=p\}.
\end{aligned}
$$

The proof that $\Pr\{\varepsilon_1^* \le \nu_1(0,\tilde{x})|\varepsilon_2^* > p\} = \Pr\{Y=1|D=0, X=\tilde{x}, P=p\}$ for all $(\tilde{x}, p) \in \operatorname{supp}(X,P)$ follows from an analogous argument.

Finally, we argue that $f_{1,2}^*$ satisfies claim (C). Consider $\Pr\{\varepsilon_1^* \le \nu_1(0,x)\}$. From (8), we have that $\nu_1(1,x) \le \nu_1(1,x_0^l(x))$ or $\nu_1(1,x) \ge \nu_1(1,x_0^u(x))$. In the former case, we have that

$$
\begin{aligned}
&\Pr\{\varepsilon_1^* \le \nu_1(0,x)\} \\
&\quad = \int_0^1\int_{-\infty}^{\nu_1(0,x)} f_{1,2}^*(t_1,t_2)\,dt_1\,dt_2 \\
&\quad = \bigg\{\int_0^{\underline{p}}\bigg(\int_{-\infty}^{\nu_1(1,x_0^l(x))} f_{1,2}^*(t_1,t_2)\,dt_1 + \int_{\nu_1(1,x_0^l(x))}^{\nu_1(0,x)} f_{1,2}^*(t_1,t_2)\,dt_1\bigg)\,dt_2 \\
&\qquad + \int_{\underline{p}}^1\int_{-\infty}^{\nu_1(0,x)} f_{1,2}^*(t_1,t_2)\,dt_1\,dt_2\bigg\} \\
&\quad = \bigg\{\int_0^{\underline{p}}\bigg(\int_{-\infty}^{\nu_1(1,x_0^l(x))} f_{1,2}(t_1,t_2)\,dt_1 + a(t_2)\int_{\nu_1(1,x_0^l(x))}^{\nu_1(0,x)} f_{1,2}(t_1,t_2)\,dt_1\bigg)\,dt_2 \\
&\qquad + \int_{\underline{p}}^1\int_{-\infty}^{\nu_1(0,x)} f_{1,2}(t_1,t_2)\,dt_1\,dt_2\bigg\} \\
&\quad = s_0^* + \Pr\{D=0, Y=1|X=x, P=\underline{p}\} = s_0.
\end{aligned}
$$

In the latter case, it suffices to show that

$$
\int_{\underline{p}}^1\int_{-\infty}^{\nu_1(0,x)} f_{1,2}^*(t_1,t_2)\,dt_1\,dt_2 = \int_{\underline{p}}^1\int_{-\infty}^{\nu_1(0,x)} f_{1,2}(t_1,t_2)\,dt_1\,dt_2.
$$

For this purpose, it suffices to show that

$$\int_{\overline{p}}^{1} \int_{\nu_1(0,x_1^l(x))}^{\nu_1(0,x_1^u(x))} f_{1,2}^*(t_1, t_2)\, dt_1\, dt_2 = \int_{\overline{p}}^{1} \int_{\nu_1(0,x_1^l(x))}^{\nu_1(0,x_1^u(x))} f_{1,2}(t_1, t_2)\, dt_1\, dt_2,$$

since outside of this region of integration $f_{1,2}^* = f_{1,2}$. Note that

$$\begin{aligned}
&\int_{\overline{p}}^{1} \int_{\nu_1(0,x_1^l(x))}^{\nu_1(0,x_1^u(x))} f_{1,2}^*(t_1, t_2)\, dt_1\, dt_2 \\
&\quad = \int_{\overline{p}}^{1} c(t_2) \int_{\nu_1(0,x_1^l(x))}^{\nu_1(0,x)} f_{1|2}(t_1|t_2)\, dt_1\, dt_2 \\
&\qquad + \int_{\overline{p}}^{1} d(t_2) \int_{\nu_1(0,x)}^{\nu_1(0,x_1^u(x))} f_{1|2}(t_1|t_2)\, dt_1\, dt_2 \\
&\quad = \int_{\overline{p}}^{1} c(t_2) \Pr\big\{\nu_1(0, x_1^l(x)) < \varepsilon_1 < \nu_1(0, x)|t_2\big\}\, dt_2 \\
&\qquad + \int_{\overline{p}}^{1} d(t_2) \Pr\big\{\nu_1(0, x) < \varepsilon_1 < \nu_1(0, x_1^u(x))|t_2\big\}\, dt_2 \\
&\quad = \int_{\overline{p}}^{1} \Pr\big\{\nu_1(0, x_1^l(x)) < \varepsilon_1 < \nu_1(1, x_1^u(x))|t_2\big\}\, dt_2 \\
&\quad = \int_{\underline{p}}^{1} \int_{\nu_1(0,x_1^l(x))}^{\nu_1(0,x_1^u(x))} f_{1,2}(t_1, t_2)\, dt_1\, dt_2,
\end{aligned}$$

as desired. The proof that $\Pr\{\varepsilon_1^* \le \nu_1(1, x)\} = s_1$ follows from an analogous argument. *Q.E.D.*

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Manuscript received February, 2010; final revision received October, 2010.